

Liouville Quantum Gravity and KPZ

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Abstract

Consider a bounded planar domain D , an instance h of the Gaussian free field on D , with Dirichlet energy $(2\pi)^{-1} \int_D \nabla h(z) \cdot \nabla h(z) dz$, and a constant $0 \leq \gamma < 2$. The **Liouville quantum gravity measure** on D is the weak limit as $\varepsilon \rightarrow 0$ of the measures

$$\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz,$$

where dz is Lebesgue measure on D and $h_\varepsilon(z)$ denotes the mean value of h on the circle of radius ε centered at z . Given a random (or deterministic) subset X of D one can define the scaling dimension of X using either Lebesgue measure or this random measure. We derive a general quadratic relation between these two dimensions, which we view as a probabilistic formulation of the KPZ relation from conformal field theory. We also present a boundary analog of KPZ (for subsets of ∂D). We illustrate (via heuristics and announced results) the connection between discrete and continuum quantum gravity and provide a framework for understanding Euclidean scaling exponents via quantum gravity.

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“There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude[s] are given by the sums over all possible surfaces with fixed boundary.” (A.M. Polyakov, Moscow, 1981.) [Pol81a]

1 Introduction

1.1 Overview

The study of certain natural probability measures on the space of two dimensional Riemannian manifolds (and singular limits of these manifolds) is often called “two-dimensional quantum gravity.” These models have been very thoroughly studied in the physics literature, in part because of connections to string theory and conformal field theory [Pol81a, Pol81b, Pol87a, Pol89, Sei90, GM93, Dav94, Dav95, AJW95, AW95, DFGZJ95, Kle95, KH96, ADJ97, Eyn01, Dup06], and to random matrix theory and geometrical models; see, e.g., the references [BIPZ78, ADF85, KKM85, Dav85, BKKM86a, BKKM86b, Kaz86, DK88a, DK90, GK89, Kos89a, Kos89b, MSS91, KK92, EZ92, Dau95, EK95, KH95, BDKS95, AAMT96, Dup98, Dup99a, Dup99b, Dup99c, EB99, KZJ99, Kos00, Dup00, DFGG00, Dup04]. More recently, a purely combinatorial approach to discretized quantum gravity has been successful [Sch98, BS02, BDFG02, AS03, BDFG03a, BDFG03b, DFG05, BDFG07, Mie07, LG07, MM07, BG08a, MW08, Mie08, BG08b, LG08].

One of the most influential papers in this field is a 1988 work of Knizhnik, Polyakov, and Zamolodchikov [KPZ88]. Building on a 1987 work of Polyakov [Pol87b], the authors derive a relationship (the **KPZ formula**) between scaling dimensions of fields defined using Euclidean geometry and analogous dimensions defined via Liouville quantum gravity (as described earlier in [Pol81a, Pol81b]). An alternative heuristic derivation using Liouville field theory in the so-called conformal gauge was proposed shortly after [Dav88a, DK89] (see also [Tak93]). The original work by KPZ has been cited roughly a thousand times in a variety of contexts, which we will not attempt to survey here, though we mention that there have been a number of explicit calculations in Liouville field theory with matching results in the random matrix theory approach, e.g., [DO94, ZZ96, FZZ, Tes01, PT02, Kos03, Zam04, KPS04, TT06, Tes07].

The relationship in [KPZ88] has never been proved or even precisely formulated mathematically. The main goal of this work is to formulate and prove the KPZ scaling dimension relationship in a probabilistic setting.

1.2 Critical Liouville quantum gravity

The study of two dimensional random surfaces makes frequent use of the Riemann uniformization theorem, which states that every smooth simply connected Riemannian mani-

fold \mathcal{M} can be conformally mapped to either the unit disc \mathbb{D} , the complex plane \mathbb{C} , or the complex sphere $\mathbb{C} \cup \{\infty\}$. (If a manifold is not simply connected then its universal cover can be conformally mapped to one of these spaces. See, e.g., Chapter 4 of [FK92] for more exposition; see also [WGY05, JWGY05, GWY03, GY02, DLJ⁺07] for approximation algorithms and beautiful computer illustrations of these maps.) Another way to say this is that \mathcal{M} can be parameterized by points $z = x + iy$ in one of these spaces in such a way that the metric takes the form $e^{\lambda(z)}(dx^2 + dy^2)$ for some real-valued function λ . The (x, y) are called *isothermal coordinates* or *isothermal parameters* for \mathcal{M} . In most of this paper we let the parameter space be a general simply connected proper subdomain D of the plane (which, of course, is conformally equivalent to \mathbb{D}).

We remark that the existence of isothermal coordinates does not require that \mathcal{M} be smooth; for example, it can be deduced whenever \mathcal{M} can be parameterized by a simply connected planar domain in which the metric has the form $E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$ where $EG - F^2 > 0$, $E > 0$, and E , F , and G are β -Hölder continuous for some $0 < \beta < 1$ [Che55].

Length, area, and curvature are easy to express in isothermal coordinates. The length of a path in \mathcal{M} parameterized by a smooth path P in D is given by

$$\int_P e^{\lambda(s)/2} ds,$$

where ds is the Euclidean length measure on D . Given a measurable subset A of D , the integral $\int_A e^{\lambda(z)} dz$ (where dz denotes Lebesgue measure on D) is the area of the portion of \mathcal{M} parameterized by A . The function $K = -e^{-\lambda}\Delta\lambda$ (where $\Delta\lambda = \lambda_{xx} + \lambda_{yy}$ is the Laplacian operator) is called the **Gaussian curvature** of \mathcal{M} . If A is a measurable subset of the (x, y) parameter space, then the integral of the Gaussian curvature with respect to the portion of \mathcal{M} parameterized by A can be written $\int_A e^{\lambda(z)} K(z) dz = \int_A -\Delta\lambda(z) dz$ where dz denotes Lebesgue measure on D . In other words, $-\Delta\lambda$ gives the density of Gaussian curvature in the isothermal coordinate space. In particular, \mathcal{M} is flat if and only if λ is harmonic.

The above suggests that one can study random simply connected Riemannian manifolds by studying random functions λ on \mathbb{C} or $\mathbb{C} \cup \{\infty\}$ or any fixed simply connected subdomain D of \mathbb{C} . In the probabilistic formulation of the so-called critical Liouville quantum gravity, λ is taken to be a multiple of the Gaussian free field (GFF), although some care will be required to make sense of this construction, since the GFF is a distribution and not a function. (The relationship between our probabilistic formulation and the original formulation of Polyakov will be discussed in Section 2.)

For concreteness, let h be an instance of a centered GFF on a bounded simply connected domain D with zero boundary conditions. This means that $h = \sum_n \alpha_n f_n$ where the α_n are i.i.d. zero mean unit variance normal random variables and the f_n are an orthonormal basis, with respect to the inner product

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

of the Hilbert space closure $H(D)$ of the space $H_s(D)$ of C^∞ real-valued functions compactly supported on D . Although this sum diverges pointwise almost surely, it does converge almost surely in the space of distributions on D , and one can also make sense of the mean value

of h on various sets. (See [She07] for a detailed account of this construction of the GFF; see Section 3.1 for a quick overview. Note that the $(2\pi)^{-1}$ in the definition above does not appear, e.g., in [She07]; including this factor in the definition, as is common in the physics literature, is equivalent to multiplying the corresponding h by $\sqrt{2\pi}$. This will simplify some of our formulas later on. In particular, in this formulation the two point covariance scales like $-\log(|z - w|)$ instead of $-(2\pi)^{-1}\log(|z - w|)$; see Section 3.1.)

Given an instance h of the Gaussian free field on D , let $h_\varepsilon(z)$ denote the mean value of h on the circle of radius ε centered at z (where $h(z)$ is defined to be zero for $z \in \mathbb{C} \setminus D$). This is almost surely a locally Hölder continuous function of (ε, z) on $(0, \infty) \times D$ (see Section 3.1). For each fixed ε , consider the surface \mathcal{M}_ε parameterized by D with metric $e^{\gamma h_\varepsilon(z)}(dx^2 + dy^2)$. We would like to define a surface \mathcal{M} parameterized by D to be some sort of limit as $\varepsilon \rightarrow 0$ of these surfaces. Since we would not expect the limit to be a Riemannian manifold in any classical sense, we have to state carefully what we mean by this. There are many ways we could attempt to make sense of this limit, depending on what quantities we focus on. For example, we could consider

1. The length of the shortest path connecting a fixed pair of points in D .
2. The area of a fixed subset of D .
3. The length of a fixed smooth curve in D .
4. The length of a smooth boundary arc of D (which becomes interesting when h is an instance of the GFF with free boundary conditions).

Intuitively, we might expect each quantity above to scale like a random constant times a (possibly different) power of ε as ε tends to zero — i.e., we would expect that if the \mathcal{M}_ε were rescaled by the appropriate powers of ε , the above quantities would have limits as $\varepsilon \rightarrow 0$. Focusing on lengths of shortest paths, one might guess that the random surfaces \mathcal{M}_ε (rescaled by some power of ε) would almost surely converge (in some natural topology on the set of metric spaces) to a non-trivial random metric space parameterized by D . However, this is not something we are currently able to prove. Focusing on areas, one might expect that for some α the renormalized area measures $\varepsilon^\alpha e^{\gamma h_\varepsilon(z)} dz$ would almost surely converge weakly to a random measure on D . This is the limit we will construct and work with in this paper. We will also address the lengths of fixed curves and boundary curves; see Section 6. Although the constructions are quite similar, we will not use the so-called Wick normal ordering terminology in this paper (see e.g., [Sim74]). We present a self-contained proof of the following (although similar measures have appeared much earlier, and are called the *Høegh-Krohn model* [HK71] — see also [AGHK79, AHK74] for a discussion on the level of Schwinger functions, and a more recent survey [AHKPS92]):

Proposition 1.1. *Fix $\gamma \in [0, 2)$ and define h and D as above. Then it is almost surely the case that as $\varepsilon \rightarrow 0$ along powers of two, the measures $\mu_\varepsilon := \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$ converge weakly to a limiting measure, which we denote by $\mu = \mu_h = e^{\gamma h(z)} dz$. This remains true if we replace h with a non-centered GFF on D — i.e., if we set $h = \bar{h} + h^0$ where \bar{h} is the zero boundary GFF on D and h^0 is a deterministic, non-zero continuous function on D .*

For each $z \in D$, denote by $C(z; D)$ the conformal radius of D viewed from z . That is, $C(z; D) = |\phi'(z)|^{-1}$ where $\phi : D \rightarrow \mathbb{D}$ is a conformal map to the unit disc with $\phi(z) = 0$. The following gives an equivalent definition of μ .

Proposition 1.2. *Write $h = \bar{h} + h^0$ where \bar{h} is the zero boundary GFF on D and h^0 is a deterministic continuous function on D . Let f_1, f_2, \dots be an orthonormal basis for $H(D)$ comprised of continuous functions on D and let h^n be the expectation of h given its projection onto the span of $\{f_1, f_2, \dots, f_n\}$. (In other words, h^n is h^0 plus the projection of \bar{h} onto the span of $\{f_1, f_2, \dots, f_n\}$.) Then $\mu = \mu_n$ (as defined in Proposition 1.1) is almost surely the weak limit for $n \rightarrow +\infty$ of the measures*

$$\mu^n = \exp \left(\gamma h^n(z) - \frac{\gamma^2}{2} \text{Var} h^n(z) + \frac{\gamma^2}{2} \log C(z; D) \right) dz. \quad (1)$$

For each measurable $A \subset D$, we have $\mathbb{E}[\mu(A)|h^n] = \mu^n(A)$. In particular,

$$\mathbb{E}\mu(A) = \int_A C(z; D)^{\frac{\gamma^2}{2}} e^{\gamma h^0(z)} dz.$$

Intuitively, we interpret the pair (D, μ) as describing a “random surface” \mathcal{M} parameterized conformally by D , with area measure given by μ . The term “random metric” is often used as well; however, we stress that, since we have not endowed D with a two point distance function, “random metric” in the Liouville quantum gravity context does not mean “random metric space.”

1.3 Scaling exponents and KPZ

Definition 1.3. *For any fixed measure μ on D (which we call the “quantum” measure), we let $B^\delta(z)$ be the Euclidean ball centered at z whose radius is chosen so that $\mu(B^\delta(z)) = \delta$. (If there does not exist a unique δ with this property, take the radius to be $\sup\{\varepsilon : \mu(B_\varepsilon(z)) \leq \delta\}$.) We refer to $B^\delta(z)$ as the **isothermal quantum ball** of area δ centered at z . In particular, if $\gamma = 0$ then μ is Lebesgue measure and $B^\delta(z)$ is $B_\varepsilon(z)$ where $\delta = \pi\varepsilon^2$.*

Given a subset $X \subset D$, we denote the ε neighborhood of X by

$$B_\varepsilon(X) = \{z : B_\varepsilon(z) \cap X \neq \emptyset\}.$$

We also define the **isothermal quantum δ neighborhood** of X by

$$B^\delta(X) = \{z : B^\delta(z) \cap X \neq \emptyset\}.$$

Translated into probability language, the so-called **KPZ formula** is a quadratic relationship between the expectation fractal dimension of a random subset of D defined in terms of Euclidean measure (which is the Liouville gravity measure with $\gamma = 0$) and the corresponding expectation fractal dimension of X defined in terms of Liouville gravity with $\gamma \neq 0$.

Fix $\gamma \in [0, 2)$ and let μ_0 denote Lebesgue measure on D . We say that a (deterministic or random) fractal subset X of D has **Euclidean expectation dimension** $2 - 2x$ and **Euclidean scaling exponent** x if the expected area of $B_\varepsilon(X)$ decays like $\varepsilon^{2x} = (\varepsilon^2)^x$, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}\mu_0(B_\varepsilon(X))}{\log \varepsilon^2} = x.$$

We say that X has **quantum scaling exponent** Δ if when X and μ (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu(B^\delta(X))}{\log \delta} = \Delta.$$

(Section 7 will provide some discrete quantum gravity heuristics that motivate the idea of taking X and D independent of one another, as well as our particular definition of scaling exponent.)

The following is the KPZ scaling exponent relation. To avoid boundary technicalities, we restrict attention here to a compact subset of D . The case of boundary exponents will be dealt with in Section 6.

Theorem 1.4. *Fix $\gamma \in [0, 2)$ and a compact subset \tilde{D} of D . If $X \cap \tilde{D}$ has Euclidean scaling exponent $x \geq 0$ then it has quantum scaling exponent Δ , where Δ is the non-negative solution to*

$$x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta. \quad (2)$$

It also turns out that Theorem 1.4 admits the following straightforward generalization:

Theorem 1.5. *Let \mathcal{X} be any random measurable subset of the set of all balls of the form $B_\varepsilon(z)$ for $\varepsilon > 0$ and z in a fixed compact subset \tilde{D} of D . Fix $\gamma \in [0, 2)$. Then if*

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E} \mu_0\{z : B_\varepsilon(z) \in \mathcal{X}\}}{\log \varepsilon^2} = x,$$

then it follows that, when \mathcal{X} and μ (as defined above) are chosen independently, we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu\{z : B^\delta(z) \in \mathcal{X}\}}{\log \delta} = \Delta,$$

where Δ is the non-negative solution to

$$x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta.$$

(Note that expectation in the above theorem is with respect to both random variables, X and μ .) We obtain Theorem 1.4 as a special case of Theorem 1.5 by writing $\mathcal{X} = \{B_\varepsilon(z) : B_\varepsilon(z) \cap X \neq \emptyset\}$. Theorem 1.5 allows us to consider x that are greater than 1 (in which case the “dimension” $2 - 2x$ would be negative). If one considers, for example, a conformal loop ensemble on D with $\kappa = 6$ (corresponding to a scaling limit of the cluster-boundary loops in site percolation on the triangular lattice) one could let \mathcal{X} be the set of balls contained in \tilde{D} that intersect ℓ distinct “macroscopic” loops (where “macroscopic” means that their diameters are greater than some fixed constant). In this case, the value x depends on ℓ and is called a **multi-arm exponent** [SD87, Dup99a, ADA99, SW01] and we may view the corresponding Δ as a quantum analog of such an exponent.

As another example, for some integer L fix distinct points z_1, z_2, \dots, z_L in $D \setminus \tilde{D}$ and run L independent Brownian motions started at the points z_1, \dots, z_L . Then let \mathcal{X} be the set of

balls $B_\varepsilon(z)$ contained in \tilde{D} with the property that the Brownian motions — stopped at the first time they intersect $\partial B_\varepsilon(z)$ — do not intersect one another.

In this case, the Euclidean scaling exponent $x = x_L$ is called a **Brownian intersection exponent** intersection. It was conjectured in [DK88b] and rigorously derived in a celebrated series of papers by Lawler, Schramm, and Werner using the Schramm-Loewner evolution with $\kappa = 6$ [LSW01a, LSW01b, LSW02]:

$$x_L = \frac{1}{24}(4L^2 - 1).$$

Although we will not fully explain this in this paper, there is a close connection between SLE_κ and Liouville quantum gravity models with $\gamma = \sqrt{\min\{\kappa, 16/\kappa\}}$ (see Section 8), in agreement with the relationship between CFT central charge c and parameter γ in Liouville quantum gravity [KPZ88, Dav88b, DK89, Sei90, GM93]. Taking $\gamma = \sqrt{16/6} = \sqrt{8/3}$ and x_L as above, the KPZ formula gives

$$\Delta_L = \frac{1}{2} \left(L - \frac{1}{2} \right),$$

which is an affine function of L . The first co-author predicted several years ago, based on an approach via discrete quantum gravity models, that this Δ would be an affine function of L (see [Dup98, Dup99b, Dup99c, Dup04] and the discussion in Section 7). The derivation is based on a simple and general geometric argument that discrete quantum gravity exponents should be in a certain sense additive together with a heuristic connection between the discrete and the continuous models. A direct calculation via discrete graphs appears in [Dup98]. This is related to the cascade relations given earlier by Lawler and Werner using different techniques [LW99].

Two papers that build on our work (as announced and presented in talks and minicourses beginning in 2007) have already been posted online: Benjamini and Schramm cited the ideas of our paper to produce an analog of Theorem 1.4 in a one dimensional cascade model; their proof uses a Frostman measure construction in place of the large deviations construction used here, and almost sure Hausdorff dimension in place of expectation dimension [BS08]. A follow up paper [RV08] adapts the arguments of [BS08] to a class of cascade models, which was expanded to include (in a revised version) a measure based on the exponential of the Gaussian free field, like the measures we construct here.

Intuitively, one reason to expect Hausdorff-like variants of KPZ to be accessible is that the second moments (and higher moments) of the random measures are essentially trivial to compute (see Section 3.2). It might be interesting to try to derive other variants of KPZ: for example, one could try to relate the actual Minkowski or Hausdorff measure of a set, in the Euclidean sense, with some kind of expected Minkowski or Hausdorff measure in the quantum sense. We will not address these alternative formulations here. However, we will present below a picturesque formulation of KPZ in terms of box decompositions.

1.4 Statement of box formulation of KPZ

Define a **diadic square** to be a closed square (including its interior) of one of the grids $2^{-k}\mathbb{Z}^2$ for some integer k . Let μ be any measure on \mathbb{C} . For $\delta > 0$, we define a (μ, δ) **box**

S to be a dyadic square S with $\mu(S) < \delta$ and $\mu(S') \geq \delta$ where S' is the dyadic parent of S . Clearly, if a point $z \in \mathbb{C}$ does not lie on a boundary of a dyadic square—and it satisfies $\mu(\{z\}) < \delta < \mu(\mathbb{C})$ —then there is a unique (μ, δ) box containing z , which we denote by $S^\delta(z)$. Let \mathcal{S}_μ^δ be the set of all (μ, δ) boxes. The boxes in \mathcal{S}_μ^δ do not overlap one another except at their boundaries. Thus, they form a tiling of \mathbb{R}^2 (see Figures 1, 2, and 3 for an illustration of this construction on a torus).

We remark that the (μ, δ) boxes should not be confused with the dyadic boxes in the so-called δ -Calderón Zygmund decomposition of μ . Readers familiar with that decomposition may recall that while the (μ, δ) boxes are dyadic squares S with $\mu(S) < \delta \leq \mu(S')$, the δ -Calderón Zygmund boxes are dyadic squares S with $\mu(S)/\mu_0(S) > \delta \geq \mu(S')/\mu_0(S')$, where μ_0 is Lebesgue measure. Roughly speaking, the μ measure on each (μ, δ) box approximates δ , while the μ density on each Calderón Zygmund box approximates δ .

When ε is a power of 2, analogously define $S_\varepsilon(z)$ to be the dyadic square containing z with edge length ε . Likewise, define

$$S_\varepsilon(X) = \{z : S_\varepsilon(z) \cap X \neq \emptyset\},$$

$$S^\delta(X) = \{z : S^\delta(z) \cap X \neq \emptyset\}.$$

The following gives the equivalence of the scaling dimension definition when boxes are used instead of balls. (The first half is well known and easy to verify.)

Proposition 1.6. *Fix $\gamma \in [0, 2)$ and let X be a random subset of a deterministic compact subset of D . Let $N(\mu, \delta, X)$ be the number of (μ, δ) boxes intersected by X and $N(\varepsilon, X)$ the number of dyadic squares intersecting X that have edge length ε (a power of 2). Then X has Euclidean scaling exponent $x \geq 0$ if and only if*

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[\varepsilon^2 N(\varepsilon, X)]}{\log \varepsilon^2} = x,$$

or equivalently,

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[N(\varepsilon, X)]}{\log \varepsilon^2} = x - 1.$$

Similarly, X has **quantum scaling exponent** Δ if and only if when X and μ (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[\delta N(\mu, \delta, X)]}{\log \delta} = \Delta, \tag{3}$$

or equivalently,

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[N(\mu, \delta, X)]}{\log \delta} = \Delta - 1.$$

Of course, this immediately implies the following restatement of Theorem 1.4 in terms of boxes instead of balls:

Corollary 1.7. *Fix $\gamma \in [0, 2)$ and a compact subset \tilde{D} of D and X and μ as above. Then if*

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[N(\varepsilon, X)]}{\log \varepsilon^2} = x - 1.$$

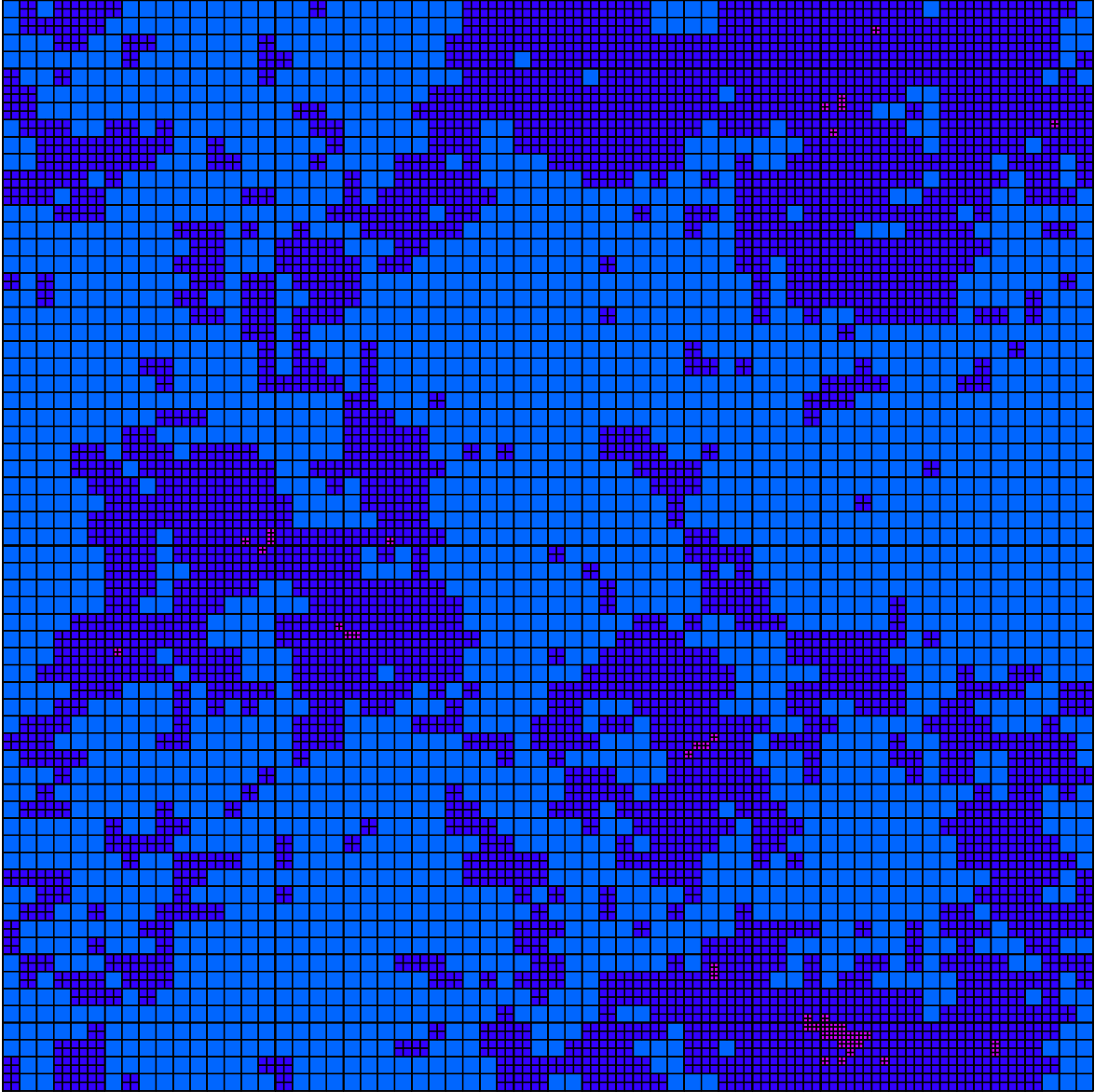


Figure 1: (μ, δ) boxes of the random measure $\mu = e^{\gamma h} dz$, where $\gamma = .5$ and h is the (discrete) Gaussian free field on a very fine (1024×1024) grid on the torus, dz is counting measure on the vertices of that grid, and δ is 2^{-12} times the total mass of μ . (We view μ as an approximation of the continuum Liouville quantum gravity measure.) One way to construct this figure is to view the entire torus as a square; then subdivide each square whose μ measure is at least δ into four smaller squares, and repeat until all squares have μ measure less than δ . The squares shown have roughly the same μ size — in the sense that each square has μ measure less than δ but each square's diadic parent has μ measure greater than δ .

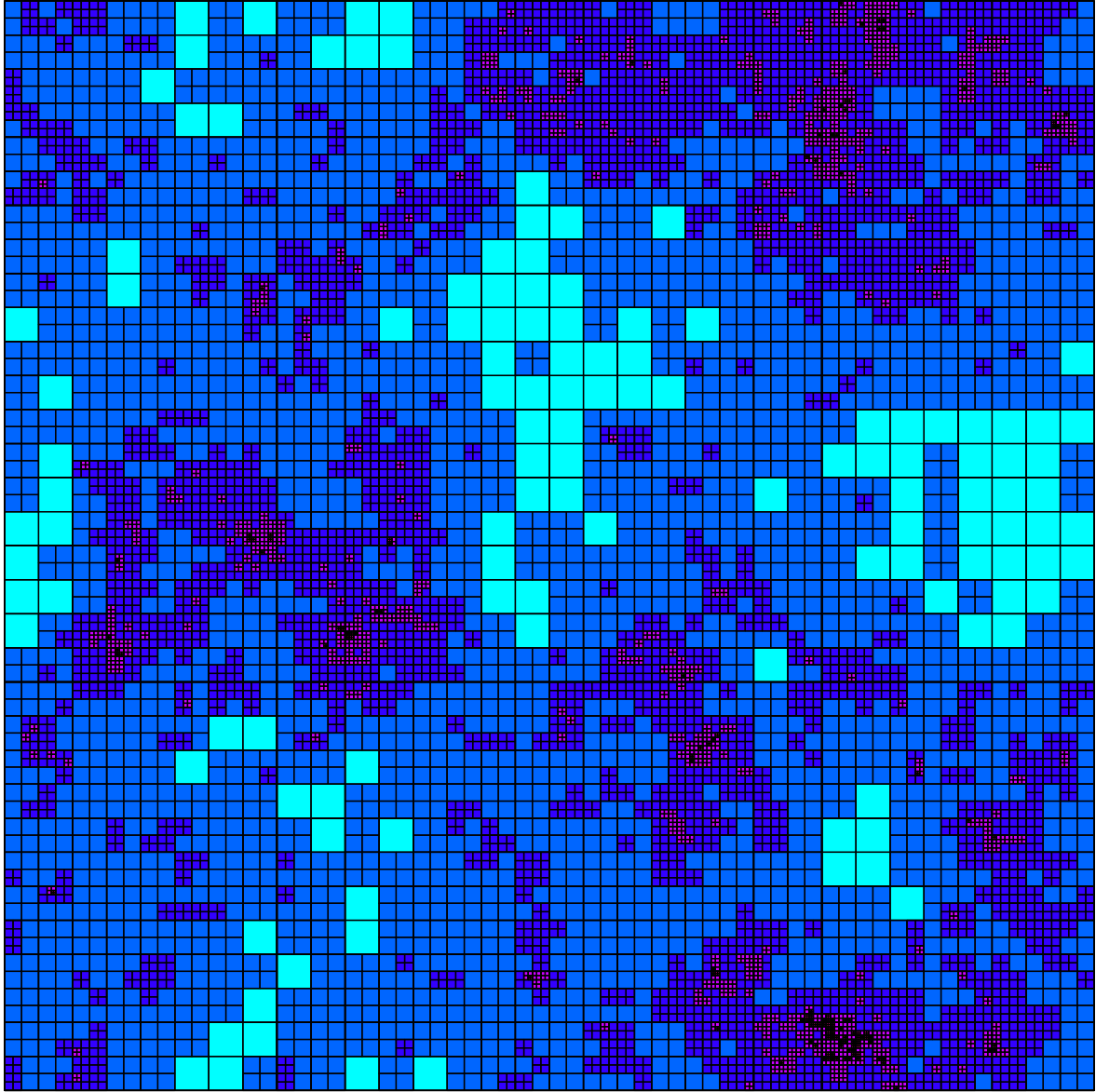


Figure 2: Analog of Figure 1 with $\gamma = 1$, using the same instance h of the GFF.

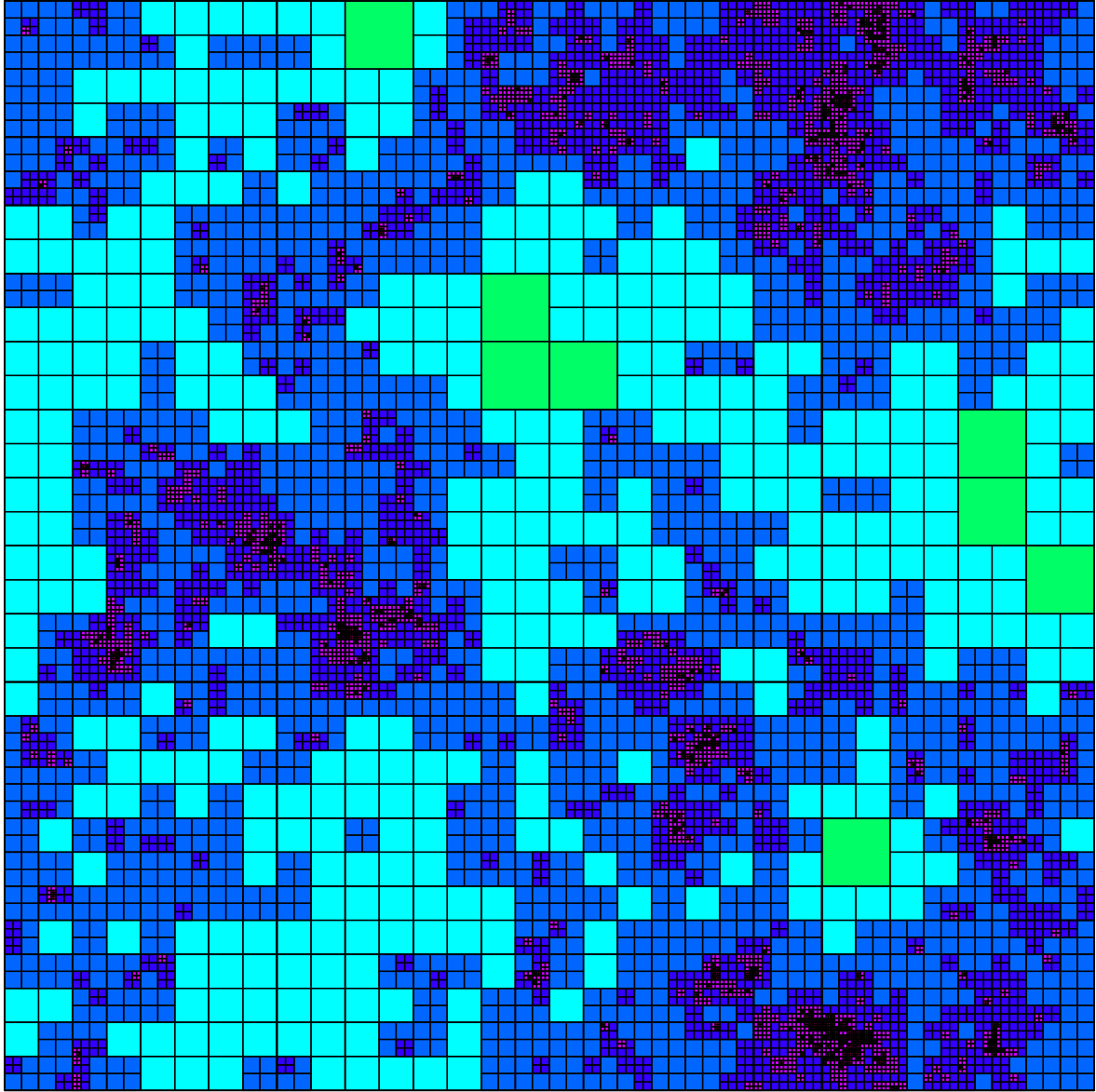


Figure 3: Analog of Figure 1 with $\gamma = 1.5$, using the same instance h of the GFF.

for some $x > 0$ then

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[N(\mu, \delta, X)]}{\log \delta} = \Delta - 1,$$

where Δ is the non-negative solution to (2).

One could also phrase Theorem 1.5 in terms of boxes instead of balls, but for simplicity we will refrain from doing this here.

2 Coordinate changes and the physical Liouville action

Polyakov understood early on that the Liouville quantum gravity action becomes a free field action in the conformal gauge, but he did not construct the random metric the way we do. In [Pol87b], where Polyakov begins the KPZ derivation, he refers to the Liouville quantum gravity action and writes

“The most simple form this formula takes is in the conformal gauge, where $g_{ab} = e^\varphi \delta_{ab}$ where it becomes a free field action. Unfortunately this simplicity is an illusion. We have to set a cut-off in quantizing this theory, such that it is compatible with general covariance. Generally, it is not clear how to do this. For that reason, we take a different approach.”

Indeed, the actual derivation given in [Pol87b] and subsequently in Knizhnik, Polyakov, and Zamolodchikov [KPZ88] is more complicated than ours and is not based on the Gaussian free field. It does not give precise mathematical meaning to the random surfaces. We feel that the Gaussian free field based random metric we construct is the correct one, at least in the sense that it is likely to arise as a scaling limit of the discrete quantum gravity models mentioned in [KPZ88] (see Section 7). In a way our approach is more similar to the work of David [Dav88a] and of Distler and Kawai [DK89], which heuristically derived KPZ from Liouville field theory in the so-called conformal gauge.

In this section, we describe how the Liouville quantum gravity measure we construct transforms covariantly under coordinate changes and use this to explain the connection between the Gaussian free field and the more familiar and more general curvature-based definition of the Liouville action that is conventional in the physics literature. The covariance properties of the random metrics in our point of view are very simple and agree with those postulated in the physics literature.

If ϕ is a conformal map from D to a domain \tilde{D} and h is a distribution on D , then we define the pullback $h \circ \phi^{-1}$ of h to be a distribution on \tilde{D} defined by $(h \circ \phi^{-1}, \tilde{\rho}) = (h, \rho)$ whenever $\rho \in H_s(D)$ and $\tilde{\rho} = |\phi'|^{-2} \rho \circ \phi^{-1}$. (Here ϕ' is the complex derivative of ϕ , and (h, ρ) is the value of the distribution h integrated against ρ .) Note that if h is a continuous function (viewed as a distribution via the map $\rho \rightarrow \int_D \rho(z)h(z)dz$), then the distribution $h \circ \phi^{-1}$ thus defined is the ordinary composition of h and ϕ^{-1} (viewed as a distribution).

The following transformation rule is a simple consequence of Proposition 1.2 and the definitions above.

Proposition 2.1. *Let h be an instance of the GFF on D and ψ a conformal map from a domain \tilde{D} to D . Write \tilde{h} for the distribution on \tilde{D} given by $h \circ \psi + Q \log |\psi'|$ where*

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

Then μ_h is almost surely the image under ψ of the measure $\mu_{\tilde{h}}$ on \tilde{D} . That is, $\mu_{\tilde{h}}(A) = \mu_h(\psi(A))$ for each Borel measurable $A \subset \tilde{D}$.

Proof. Using the notation of Proposition 1.2, if f_1, f_2, \dots are an orthonormal basis for $H(D)$, then the conformal invariance of $(\cdot, \cdot)_{\nabla}$ implies that $f_1 \circ \psi, f_2 \circ \psi, \dots$ are an orthonormal basis for $H(\tilde{D})$, and as $n \rightarrow \infty$ the functions $h^n \circ \psi$ converge in law to the GFF on \tilde{D} , and the functions $\tilde{h}^n = h^n \circ \psi + Q \log |\psi'|$ converge in law to \tilde{h} . If we define $\tilde{\mu}^n$ analogously to μ^n in (1) but with h^n replaced by \tilde{h}^n , then the $\tilde{\mu}^n$ converge weakly to the random distribution $\tilde{\mu} := \mu_{\tilde{h}}$.

To see that μ is the image of $\tilde{\mu}$ under ψ , we will observe that μ^n is the image of $\tilde{\mu}^n$ under ψ for each n . To see this, consider the term $Q \log |\psi'| = (2/\gamma) \log |\psi'| + (\gamma/2) \log |\psi'|$ in the definition of \tilde{h} . Adding $(2/\gamma) \log |\psi'|$ to $h^n \circ \psi$ corresponds to multiplying (1) by a factor of $|\psi'|^2$. This compensates for the fact that the Radon-Nikodym derivative of a measure on \tilde{D} at a point z and the Radon-Nikodym derivative of the same measure pushed forward on D at $\psi(z)$ differ by a factor of $|\psi'(z)|^2$. Adding $(\gamma/2) \log |\psi'(z)|$ to $h^n \circ \psi$ compensates the expression (1) for the change in conformal radius: $\log C(\psi(z); D) - \log C(z; \tilde{D}) = \log |\psi'(z)|$. \square

We interpret Proposition 2.1 as a rule for changing the parametrization of a random metric. For example, consider the random metric one gets by taking D to be some fixed domain. Then if we are given any other domain \tilde{D} and a conformal map $\psi : \tilde{D} \rightarrow D$, we may wish to consider the same random metric parameterized by \tilde{D} instead of D . In this case, the transformation rule tells us that on \tilde{D} we should consider the Liouville quantum gravity measure defined using $\tilde{h} = h \circ \psi + Q \log |\psi'|$, where $h \circ \psi$ is the GFF on \tilde{D} with zero boundary conditions.

We remark that one can make a similar argument when \tilde{D} is a curved simply connected manifold and $\psi : \tilde{D} \rightarrow D$ a conformal map; in this case, the function λ can be defined as follows: if the metric on \tilde{D} — when parameterized by D using the map ψ^{-1} — takes the form $e^{\lambda(z)}(dx^2 + dy^2)$, for $z \in D$, then we write $\lambda(\psi(w)) = -2 \log |\psi'(w)|$ for $w \in \tilde{D}$. Although we will not prove it here, the analog of Proposition 1.1 for smooth curved surfaces is straightforward, and the transformation rule Proposition 2.1 remains the same in this case; as in the flat case, the law of the Liouville quantum gravity measure on D pulled back to \tilde{D} is that of $\tilde{h} = h \circ \psi + Q \log |\psi'|$ where $h \circ \psi$ is the GFF on \tilde{D} with zero boundary conditions. (Alternatively, we may take this as a definition of the Liouville quantum gravity measure on curved \tilde{D} with zero boundary conditions.)

The remainder of this subsection describes the connection between our notation and the common physics literature Liouville gravity notation. (This discussion can be skipped, on a first read, by readers with no prior familiarity with the latter.) What we call the GFF on D (with the $1/2\pi$ normalization, as discussed in the introduction) is often written (sometimes without a rigorous definition) as the measure $e^{-S(h)}dh$, where

$$S(h) = \frac{1}{4\pi} \int_D \nabla h(z) \cdot \nabla h(z) dz$$

is called the **action** and dh is defined heuristically as a “uniform measure on the space of all functions.” (Of course, the latter makes perfect sense if one considers only a finite dimensional vector space of functions, such as real-valued functions defined on the vertices of a lattice, or functions whose Fourier coefficients beyond a certain frequency threshold are identically zero—in this case dh would be the Lebesgue measure on the vector space.) In this paper, we will write

$$(h, h)_\nabla := \frac{1}{2\pi} \int_D \nabla h(z) \cdot \nabla h(z) dz,$$

so that the above becomes $S(h) = \frac{1}{2}(h, h)_\nabla$.

In the following, let D be a subdomain of \mathbb{C} and \tilde{D} a possibly curved surface for which there is a conformal map $\psi : \tilde{D} \rightarrow D$. Write $\tilde{h}^0 = \log |\psi'|$. Now, if we switch parametrization to \tilde{D} , we are adding $Q\tilde{h}^0$ deterministically to $h \circ \psi$ to get \tilde{h} , so we may rewrite the action as

$$S = \frac{1}{2}(\tilde{h} - Q\tilde{h}^0, \tilde{h} - Q\tilde{h}^0)_\nabla,$$

which (at least when \tilde{h}^0 is smooth and compactly supported) is seen by integrating by parts to be equivalent (up to the additive constant $\frac{1}{2}\|Q\tilde{h}^0\|_\nabla^2$) to

$$S = \frac{1}{4\pi} \int_{\tilde{D}} dw \left(\nabla \tilde{h}(w) \cdot \nabla \tilde{h}(w) + 2\tilde{h}(w)Q\Delta \tilde{h}^0(w) \right), \quad (4)$$

where the pairing $\nabla \tilde{h}(w) \cdot \nabla \tilde{h}(w)$ and the Laplacian $\Delta \tilde{h}^0(w)$ are now defined using the metric on \tilde{D} and where now dw represents the measure on \tilde{D} instead of D . This can also be written

$$S = \frac{1}{4\pi} \int_{\tilde{D}} dw \left(\nabla \tilde{h}(w) \cdot \nabla \tilde{h}(w) + Q\tilde{h}(w)K(w) \right), \quad (5)$$

where K is the Gaussian curvature of \tilde{D} and dw is integration with respect to that metric. (When \tilde{h}^0 is not compactly supported, the formula can be modified to include a term for boundary curvature, but we will not discuss this here.)

Adding in one additional term which is a constant μ times the total area of \tilde{D} (and making the following symbol substitutions: $b = \gamma/2$, $\varphi = \tilde{h}$, g is the underlying metric of \tilde{D} , and j and k are summed-over indices ranging over the two tangent space directions), we obtain the more familiar formula for the Liouville action:

$$S = \frac{1}{4\pi} \int_{\tilde{D}} dw \sqrt{g} \left(g^{jk} \partial_j \varphi \partial_k \varphi + QK\varphi + 4\pi\mu e^{2b\varphi} \right),$$

where $Q = b + b^{-1}$. The action is defined similarly when free boundary conditions are used instead of zero boundary conditions — or when \tilde{D} is a compact Riemann surface of some genus. (In this case, $e^{-S(\varphi)}d\varphi$ is an infinite measure, although it can be “localized,” e.g., by requiring the mean value of φ to be zero.)

This paper will focus exclusively on the case $\gamma \in [0, 2)$ (which is said to correspond to physical models below the central charge $c = 1$ threshold) and $\mu = 0$ (the so-called **critical** Liouville quantum gravity). The string theory and quantum gravity literatures deal with other parameter choices as well — including non-zero μ and complex values for γ and Q — but these appear to be beyond the scope of our methodology, in part because, when S is complex valued, the expression $e^{-S(\varphi)}d\varphi$ is no longer a probability measure in even a heuristic sense.

3 Constructing the random measures

3.1 GFF definition and normalization

Let D be a bounded planar domain and let dz denote Lebesgue measure on D . We assume the reader is familiar with the Gaussian free field, as defined, e.g., in [She07], but we briefly review the definition here. As described earlier, to make our formulas consistent with the physics literature, the definitions of Green's function and the Dirichlet form will differ from the ones in [She07] by factors of 2π .

Let $H_s(D)$ be the space of C^∞ real-valued functions compactly supported on D . We define the Dirichlet inner product

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

on $H_s(D)$. Then an instance h of the Gaussian free field (GFF) may be viewed as a standard Gaussian on the Hilbert space closure $H(D)$ of $H_s(D)$ (i.e., as a sum of the form $\sum_n \alpha_n f_n$ where f_n are any orthonormal basis for $H(D)$) — the sum converges almost surely in the space of distributions on D , see [She07]. In fact, we may define $(h, f)_\nabla$ as random variables for non-smooth f as well; these are zero mean Gaussian random variables for each $f \in H(D)$, and

$$\text{Cov}((h, f_1)_\nabla, (h, f_2)_\nabla) = (f_1, f_2)_\nabla.$$

The collection of random variables $(h, f)_\nabla$ for $f \in H(D)$ is thus a Hilbert space (isomorphic to $H(D)$) under the covariance inner product.

When $x \in D$ is fixed, we let $\tilde{G}_x(y)$ be the harmonic extension to D of the function on ∂D given by $-\log|y - x|$. Then **Green's function in the domain D** is defined by

$$G(x, y) = -\log|y - x| - \tilde{G}_x(y).$$

When $x \in D$ is fixed, Green's function may be viewed as a distributional solution of $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$ with zero boundary conditions [She07]. It is non-negative for all $x, y \in D$.

For any function ρ on $H_s(D)$, we write $\Delta^{-1}\rho$ for the function

$$-\frac{1}{2\pi} \int_D G(\cdot, y) \rho(y) dy.$$

Integration by parts shows that this is a C^∞ (though not necessarily compactly supported) function in D whose Laplacian is ρ . We use the same notation for more general measurable functions ρ , as well as the case that ρ is a measure. (For example, we will sometimes speak of the inverse Laplacian of uniform measure on a particular circle or disc contained in D .)

If $f_1 = -\Delta^{-1}\rho_1$ and $f_2 = -\Delta^{-1}\rho_2$, then integration by parts implies that $(f_1, f_2)_\nabla = (2\pi)^{-1}(\rho_1, -\Delta^{-1}\rho_2)$, where (\cdot, \cdot) denotes the standard inner product on $L^2(D)$. We next observe that every $h \in H(D)$ is naturally a distribution, since we may define the map (h, \cdot) by $(h, \rho) := 2\pi(h, -\Delta^{-1}\rho)_\nabla$. (It is not hard to see that $-\Delta^{-1}\rho \in H(D)$, since its Dirichlet energy is given explicitly by (6).) When $-\Delta f = \rho$, we may write $(h, \rho) = 2\pi(h, f)_\nabla$, and hence

$$\text{Cov}((h, \rho_1), (h, \rho_2)) = (2\pi)^2(f_1, f_2)_\nabla.$$

We claim that the latter expression may be rewritten to give

$$\text{Cov}((h, \rho_1), (h, \rho_2)) = \int_{D \times D} \rho_1(x) G(x, y) \rho_2(y) \, dx \, dy \quad (6)$$

where $G(x, y)$ is Green's function in D . Since $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$ and

$$\int_D G(x, y) \rho_2(y) \, dy = -2\pi\Delta^{-1}\rho_2(x),$$

we obtain (6) by multiplying each side by $-\Delta f_1(x) = \rho_1(x)$ and integrating by parts with respect to x .

Denote by $h_\varepsilon(z)$ the average value of h on the circle of radius ε centered at z . Similar averages were considered in [Bau90]. (For this definition, we assume h is identically zero outside of D .) Then $h_\varepsilon(z)$ is a Gaussian process with covariances defined by

$$G_{\varepsilon_1, \varepsilon_2}(z_1, z_2) := \text{Cov}(h_{\varepsilon_1}(z_1), h_{\varepsilon_2}(z_2)) \quad (7)$$

given by

$$\int G(x, y) \rho_{\varepsilon_1}^{z_1}(x) \rho_{\varepsilon_2}^{z_2}(y) \, dx \, dy$$

where $\rho_\varepsilon^z(x)dx$ is the uniform measure (of total mass one) on $\partial B_\varepsilon(z)$. In fact (like Brownian motion) the process $h_\varepsilon(z)$ determines a random continuous function (of z and ε):

Proposition 3.1. *The process $h_\varepsilon(z)$ has a modification which is almost surely locally η -Hölder continuous in the pair $(z, \varepsilon) \in \mathbb{C} \times (0, \infty)$ for every $\eta < 1/2$.*

In other words, the Hölder regularity enjoyed by $h_\varepsilon(z)$ — as a function of the pair (ε, z) — is the same as that of Brownian motion or the Brownian sheet. In fact, as we observe below (Proposition 3.3), when z is fixed, h_ε is a Brownian motion with respect to the parameter $t = -\log \varepsilon$. We may view h_ε as an approximation to h that gets better as $\varepsilon \rightarrow 0$. Before we prove Proposition 3.1, let us make some observations about the covariance function $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2)$ defined in (7). (We will also sometimes write $G_\varepsilon(z_1, z_2) := G_{\varepsilon, \varepsilon}(z_1, z_2)$.)

First we define the function $\xi_\varepsilon^z(y)$, for $y \in D$, to be equal to $-\log \max(\varepsilon, |z - y|)$ minus the harmonic extension $\tilde{G}_z(y)$ to D of the restriction of $-\log \max(\varepsilon, |z - y|)$ to ∂D . Observe that this $\xi_\varepsilon^z(y)$ tends to zero as $y \rightarrow \partial D$ and that as a distribution $-\Delta \xi_\varepsilon^z$ (restricted to D) is equal to $2\pi\rho_\varepsilon^z$, where as before ρ_ε^z is a uniform measure on $D \cap \partial B_\varepsilon(z)$. Integrating by parts, we immediately have the following:

Proposition 3.2. *The function $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2)$ is equal to the mean value of $\xi_{\varepsilon_1}^{z_1}$ on the circle $\partial B_{\varepsilon_2}(z_2)$. In particular, if $B_{\varepsilon_1}(z_1)$ and $B_{\varepsilon_2}(z_2)$ are disjoint and contained in D then $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2) = G(z_1, z_2)$. If $B_{\varepsilon_1}(z) \subset D$ and $\varepsilon_1 \geq \varepsilon_2$ then*

$$G_{\varepsilon_1, \varepsilon_2}(z, z) = -\log \varepsilon_1 + \log C(z; D).$$

Proof of Proposition 3.1. We first claim that for each ε_0 and D there exists a constant K such that

$$\text{Var}(h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) \leq K|\varepsilon_1 - \varepsilon_2| + K|z_1 - z_2|$$

for all $z_1, z_2 \in D$ and $\varepsilon_1, \varepsilon_2 \in [\varepsilon_0, \infty)$. Since the variance can only increase if D is replaced with a larger domain, it suffices to show this holds when D is replaced by a sufficiently large disc D' (say, centered in D with 10 times the diameter r of D), and ε is restricted to values in $[\varepsilon_0, 5r]$. (For larger values of ε , the set $\partial B_\varepsilon(z)$ cannot intersect D when $z \in D$.) Since

$$\text{Var}(h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) = G_{\varepsilon_1, \varepsilon_1}(z_1, z_1) - 2G_{\varepsilon_1, \varepsilon_2}(z_1, z_2) + G_{\varepsilon_2, \varepsilon_2}(z_2, z_2),$$

it suffices to show that $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2)$ is a Lipschitz function of $(\varepsilon_1, \varepsilon_2, z_1, z_2)$ for the range of $(\varepsilon_1, \varepsilon_2, z_1, z_2)$ values indicated above. This follows from Proposition 3.2 and the fact (whose proof we leave to the reader) that ξ_ε^z is a Lipschitz function when $z \in D$ and $\varepsilon > \varepsilon_0$, with a Lipschitz constant that holds uniformly over these ε and z values.

The claim implies that for some K we have

$$\mathbb{E}[|h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)|^\alpha] \leq K|z_1 - z_2|^{\alpha/2}$$

for all $\alpha > 0$. This puts us in the setting of the multiparameter Kolmogorov-Centsov theorem [KS91, PM], which states the following: Suppose that the random field $X(a)$, $a \in \prod_{i=1}^n [0, t_i]$ satisfies $\mathbb{E}[|X(a) - X(b)|^\alpha] \leq K|a - b|^{1+\beta}$ for all a, b , for some fixed constants α, β, K . Then there exists an almost surely continuous modification of the random field and this process is γ -Hölder continuous for every $\gamma < \beta/\alpha$. Applying this for large α allows us to deduce that $h_\varepsilon(z)$, as a function of ε and z , is locally η -Hölder continuous for all $\eta < 1/2$. \square

Proposition 3.3. *Write $\mathcal{V}_t = h_{e^{-t}}(z)$, and $t_0^z = \inf\{t : B_{e^{-t}}(z) \subset D\}$. If $z \in D$ is fixed, then the law of*

$$V_t := \mathcal{V}_{t_0^z+t} - \mathcal{V}_{t_0^z}$$

is a standard Brownian motion in t .

Proof. Since we already know that the $h_\varepsilon(z)$ are jointly Gaussian random variables, it is enough to compute the variances of $h_\varepsilon(z)$ and $h_{\varepsilon'}(z)$ for fixed $\varepsilon, \varepsilon'$, and these are given in Proposition 3.2. \square

3.2 Random metrics: Liouville quantum gravity

The remainder of the paper makes frequent use of the following simple fact, which the reader may recall (or verify): if N is a Gaussian random variable with mean a and variance b then

$$\mathbb{E} e^N = e^{a+b/2}. \quad (8)$$

Since $\mathbb{E} h_\varepsilon(z) = 0$ when h is an instance of the GFF with zero boundary conditions, we have

$$\mathbb{E} e^{\gamma h_\varepsilon(z)} = e^{\text{Var}[\gamma h_\varepsilon(z)]/2}.$$

Recall $G_\varepsilon(z, z) = \log C(z; D) - \log \varepsilon$ when $B_\varepsilon(z) \subset D$. Then we have

$$\mathbb{E} e^{\gamma h_\varepsilon(z)} = \exp\left(\gamma^2/2(-\log \varepsilon + \log C(z; D))\right) = \left(\frac{C(z; D)}{\varepsilon}\right)^{\gamma^2/2}. \quad (9)$$

More general moments of the random variables $e^{\gamma h_\varepsilon(z)}$ are also easy to calculate. For example, we have

$$\mathbb{E}e^{\gamma h_\varepsilon(y)}e^{\gamma h_\varepsilon(z)} = \exp(\text{Var}[\gamma(h_\varepsilon(y) + h_\varepsilon(z))]/2) = \exp\left(\frac{\gamma^2}{2}(G_\varepsilon(y, y) + G_\varepsilon(z, z) + 2G_\varepsilon(y, z))\right). \quad (10)$$

By Proposition 3.2 we have $G_\varepsilon(y, z) = G(y, z)$ whenever $|y - z| \geq 2\varepsilon$ and $B_\varepsilon(y) \cup B_\varepsilon(z) \subset D$. In this case we have

$$\mathbb{E}e^{\gamma h_\varepsilon(y)}e^{\gamma h_\varepsilon(z)} = \left(\frac{C(y; D)C(z; D)}{\varepsilon^2}\right)^{\gamma^2/2} e^{\gamma^2 G(y, z)}.$$

Write $\bar{h}_\varepsilon = \gamma h_\varepsilon + \frac{\gamma^2}{2} \log \varepsilon$. Then we have

$$\mathbb{E}e^{\bar{h}_\varepsilon(z)} = C(z; D)^{\gamma^2/2} \asymp 1$$

and when $|y - z| > 2\varepsilon$ we have

$$\mathbb{E}e^{\bar{h}_\varepsilon(y)}e^{\bar{h}_\varepsilon(z)} = (C(y; D)C(z; D))^{\gamma^2/2} e^{\gamma^2 G(y, z)} \asymp (C(y; D)C(z; D))^{\gamma^2/2} |y - z|^{-\gamma^2} \asymp |y - z|^{-\gamma^2}$$

where \asymp indicates that equality holds up to a constant factor when y and z are restricted to any compact subset of D .

Now, for each fixed ε , write $\mu_\varepsilon := e^{\bar{h}_\varepsilon(z)} dz$ (which in essence corresponds to the “Wick normal ordering” of the original measure [Sim74]). We now argue that these converge weakly to a limiting random measure on D .

Proof of Proposition 1.1. Fix $\gamma \in [0, 2)$. It is easy to see that if for each diadic square S compactly supported in D the random variables $\mu_{2^{-k}}(S)$ converge to a finite limit as $k \rightarrow \infty$, almost surely, then $\mu_{2^{-k}}$ almost surely converges weakly to a limiting measure. We will prove convergence of $\mu_{2^{-k}}(S)$ by showing that the expectation of $|\mu_{2^{-k}}(S) - \mu_{2^{-k-1}}(S)|$ decays exponentially in k . Without loss of generality, we may assume S is the unit square $[0, 1]^2$, so that $\mu_\varepsilon(S)$ is precisely the mean value of $e^{\bar{h}_\varepsilon(z)}$ on S .

As shown above, we have

$$\mathbb{E}e^{\bar{h}_\varepsilon(z)} = C(z; D)^{\gamma^2/2},$$

(which is bounded between positive constants) when $z \in S$ and ε is sufficiently small.

For $y = (y_1, y_2) \in (0, 1)^2$ and $k \geq 1$, let S_k^y be the set of 2^{2k} points $(a, b) \in S$ with the property that $(2^k a - y_1, 2^k b - y_2) \in \mathbb{Z}^2$. Let A_k^y be the mean value of $\exp \bar{h}_{2^{-k-1}}(z)$ on the set S_k^y . Let B_k^y be the mean value of $\exp \bar{h}_{2^{-k-2}}(z)$ over the same set.

Clearly, $\mu_{2^{-k-1}}(S)$ is the mean value of A_k^y over $y \in [0, 1]^2$ and $\mu_{2^{-k-2}}(S)$ the mean value of B_k^y over $y \in [0, 1]^2$. Applying Jensen’s inequality to the convex function $|\cdot|$, it now suffices for us to prove that $\mathbb{E}|A_k^y - B_k^y|$ tends to zero exponentially in k (uniformly in y). Since the balls of radius 2^{-k-1} centered at points in S_k^y do not overlap, we have that conditioned on the values of $h_{2^{-k-1}}(z)$ for $z \in S_k^y$, the random variables $h_{2^{-k-2}}(z)$ for $z \in S_k^y$ are independent of one another; each is a Gaussian of variance $\log 2$ and mean $h_{2^{-k-1}}(z)$.

Hence, given the values of $h_{2^{-k-1}}(z)$ for $z \in S_k^y$, the value of conditional variance of $A_k^y - B_k^y$ is given by

$$2^{-4k} \sum_{z \in S_k^y} \text{Var} \left(e^{\bar{h}_{2^{-k-1}}(z)} - e^{\bar{h}_{2^{-k-2}}(z)} | h_{2^{-k-1}}(z) \right) = 2^{-4k} C \sum_{z \in S_k^y} \left(e^{\bar{h}_{2^{-k-1}}(z)} \right)^2, \quad (11)$$

where

$$C = \text{Var} \left(e^{\bar{h}_{2^{-k-1}}(z)} - e^{\bar{h}_{2^{-k-2}}(z)} | h_{2^{-k-1}}(z) = 0 \right).$$

Note that C is a constant that does not depend on k and z . The unconditional variance of $A_k^y - B_k^y$ is given by the expectation of (11). It is tempting to argue that this expectation tends to zero exponentially in k (which would in turn imply that $\mathbb{E}|A_k^y - B_k^y|$ tends to zero exponentially in k), but this turns out to be true only for $0 \leq \gamma^2 < 2$ and not for $2 \leq \gamma^2 < 4$. To see this, note that

$$\mathbb{E} \left[(\varepsilon^2 e^{\bar{h}_\varepsilon(z)})^2 \right] = \varepsilon^4 \mathbb{E}[e^{2\gamma h_\varepsilon + \gamma^2 \log \varepsilon}] \asymp \varepsilon^4 e^{-\frac{4\gamma^2 \log \varepsilon}{2} + \gamma^2 \log \varepsilon} = \varepsilon^{4-\gamma^2}. \quad (12)$$

Summing over the ε^{-2} points yields (up to an ε -independent constant factor) $\varepsilon^{2-\gamma^2}$, which does not tend to zero when $\gamma^2 \geq 2$.

However, we can deal with the case $\gamma^2 \geq 2$ by breaking the sum defining $A_k^y - B_k^y$ into two parts and dealing with them separately. The idea is that the variance of $A_k^y - B_k^y$ is only large because of rare occurrences (where $h_\varepsilon(z)$ is much larger than typical) whose contribution to the expectation of A_k^y is exponentially small in k .

To make this precise, fix some α with $\gamma < \alpha < 2\gamma$ and let \tilde{S}_k^y denote set of points $z \in S_k^y$ with the property that $h_\varepsilon(z) > -\alpha \log \varepsilon$, where $\varepsilon = 2^{-k}$. Let \tilde{A}_k^y denote the mean value of $1_{\tilde{S}_k^y} \exp \bar{h}_{2^{-k-1}}(z)$ and \tilde{B}_k^y the mean value of $1_{\tilde{S}_k^y} \exp \bar{h}_{2^{-k-2}}(z)$.

We claim that $\mathbb{E}\tilde{A}_k^y$ tends to zero exponentially in k . To see this, observe that for fixed $z \in S$, the random variable $h_\varepsilon(z)$ is a centered Gaussian with variance $\sigma^2 = -\log \varepsilon$ plus a constant; and the expectation of $e^{\bar{h}_\varepsilon(z)}$ —which we know to be constant for all ε small enough so that z is distance at least ε from the boundary of D —takes the form $\int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta$. This expression is a constant times a Gaussian integral of mean $\gamma\sigma^2$ and variance σ^2 ; thus

$$\frac{\int_{\mathbb{R}} 1_{\eta > \alpha\sigma^2} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta}{\int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta}$$

is the probability that a normal random variable η of mean $\gamma\sigma^2$ and variance σ^2 satisfies $\eta > \alpha\sigma^2$, and this clearly tends to zero exponentially in σ^2 , from which the claim easily follows.

Note that by Proposition 3.3 $\mathbb{E}\tilde{B}_k^y = \mathbb{E}\tilde{A}_k^y$, and in particular $\mathbb{E}\tilde{B}_k^y$ also tends to zero exponentially in k . Since \tilde{B}_k^y and \tilde{A}_k^y are non-negative, applying Jensen's inequality to $|\cdot|$ shows that $\mathbb{E}|\tilde{B}_k^y - \tilde{A}_k^y|$ tends to zero exponentially in k .

For the next step, we wish to bound $\mathbb{E}|(B_k^y - \tilde{B}_k^y) - (A_k^y - \tilde{A}_k^y)|$, which requires us to consider points in $S_k^y \setminus \tilde{S}_k^y$ and the expectation

$$\varepsilon^4 \mathbb{E} \left[1_{\eta < \alpha\sigma^2} \left(e^{\bar{h}_\varepsilon(z)} \right)^2 \right] = \varepsilon^4 \mathbb{E}[1_{\eta < \alpha\sigma^2} e^{2\gamma h_\varepsilon + \gamma^2 \log \varepsilon}]. \quad (13)$$

This differs from (12) by a factor that represents the probability that a Gaussian random variable with variance $-\log \varepsilon$ and mean $-2\gamma \log \varepsilon$ is less than $-\alpha \log \varepsilon$. Since $\alpha < 2\gamma$, this probability decays exponentially in $-\log \varepsilon$ at rate $(2\gamma - \alpha)^2/2$. Thus (13) becomes, up to a constant factor (universal in ε and $z \in S$),

$$\varepsilon^{4-\gamma^2} \varepsilon^{(2\gamma-\alpha)^2/2}.$$

Summing over the ε^{-2} points, we obtain

$$\varepsilon^{2-\gamma^2+(2\gamma-\alpha)^2/2}.$$

To conclude, we only need to make sure we chose $\alpha \in (\gamma, 2\gamma)$ small enough so that the sum in the exponent is positive, and this is clearly possible. In fact, taking α close to γ , the exponent becomes close to $2 - \gamma^2 + \gamma^2/2 = 2 - \frac{\gamma^2}{2}$, which is positive when $\gamma < 2$. \square

3.3 Rooted random metrics

Before proving Proposition 1.2, we introduce a notion of rooted random metric and use it to prove a uniform integrability result for the random variables $\mu_\varepsilon(S)$ discussed above.

Write $\Theta_\varepsilon := Z_\varepsilon e^{\gamma h_\varepsilon(z)} dz dh$, where Z_ε is a constant chosen to make Θ_ε a probability measure. Sampling from Θ_ε may be described as a two step procedure. First sample z from its marginal distribution. Then sample h from the distribution of the Gaussian free field *weighted* by $e^{\gamma h_\varepsilon(z)}$. The latter has the law of the original GFF *plus* $\gamma \xi_\varepsilon^z$ where ξ_ε^z satisfies a Dirichlet problem: $-\Delta \xi_\varepsilon^z$ is the multiple of the uniform measure on $\partial B_\varepsilon(z)$ with total mass 2π (because h is $\sqrt{2\pi}$ times the standard GFF; if h were the standard GFF the total mass would be 1). As noted in Section 3.1, this ξ_ε^z has been computed explicitly:

$$\xi_\varepsilon^z(y) = -\log \max\{|z - y|, \varepsilon\} - \tilde{G}_z(y),$$

where \tilde{G}_z is the harmonic interpolation to D of the first term on ∂D , as long as $B_\varepsilon(z) \subset D$.

For each fixed ε , the marginal distribution of z is given by $f(z)dz$ where $f(z) = \mathbb{E}_h e^{\gamma h_\varepsilon(z)}$. Thus $f(z)$ is proportional to $e^{(\gamma^2/2) \log C(z; D)}$ by (9).

Let Θ be the limit of the measures Θ_ε as $\varepsilon \rightarrow 0$: that is, Θ is the measure on pairs (z, h) for which the marginal distribution of z is proportional to $e^{(\gamma^2/2) \log C(z; D)} dz$ and, given z , the Θ conditional law of h is that of the original GFF plus the deterministic function $\gamma \xi_0^z$ (viewed as a distribution). When $S \subset D$ we will also write Θ_ε^S for the measure Θ_ε conditioned on the event $z \in S$. The following is obvious from our definitions:

Proposition 3.4. *With Θ probability one, z is a γ -thick point of h . That is,*

$$\liminf_{\varepsilon \rightarrow 0} h_\varepsilon(z) / \log \varepsilon^{-1} \geq \gamma.$$

(In fact, the limit exists and equality holds almost surely.)

Since the Θ marginal law of h is absolutely continuous with respect to the law of h (with Radon-Nikodym derivative $\mu_h(D)$), this implies that μ_h is almost surely supported on γ -thick points. It was shown by Hu, Miller, and Peres that the set of γ -thick points has Hausdorff dimension $2 - \frac{\gamma^2}{2}$ almost surely [HMP].

Proof of Proposition 1.2. The almost sure weak convergence of the μ^n to a limit $\tilde{\mu}$ is immediate from the martingale convergence theorem. Recall the expression (1)

$$\mu^n = \exp \left(\gamma h^n(z) - \frac{\gamma^2}{2} \text{Var} h^n(z) + \frac{\gamma^2}{2} \log C(z; D) \right) dz,$$

and observe that for each z , the exponential term

$$\exp\left(\gamma h^n(z) - \frac{\gamma^2}{2}\text{Var}h^n(z) + \frac{\gamma^2}{2}\log C(z; D)\right)$$

is a non-negative martingale with respect to the filtration of h^n . (This is a consequence of (8).) Fubini's theorem implies that $\mu^n(A)$ is a martingale for any Borel measurable set $A \subset D$, and the martingale convergence theorem implies that the limit $\lim \mu^n(A)$ exists almost surely. In particular, this holds whenever A is a diadic square contained in D and from this easily follows the desired weak convergence.

We still need to show that $\tilde{\mu} = \mu$ almost surely, where μ is as constructed in Proposition 1.1. Let h_ε^n denote the mean value of h^n on $\partial B_\varepsilon(z)$. For each particular choice of z , and ε small enough so that $B_\varepsilon(z) \subset D$, and for each n , we have

$$\mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon^n} | h^n] = \exp\left(\gamma h_\varepsilon^n(z) - \frac{\gamma^2}{2}\text{Var}h_\varepsilon^n(z) + \frac{\gamma^2}{2}\log C(z; D)\right).$$

Taking the limit as $\varepsilon \rightarrow 0$ and using the continuity of $h^n(z)$ and $\text{Var}h^n(z)$ and the expression (1), we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S) | h^n] = \mu^n(S) \quad (14)$$

for each diadic square S .

We next wish to argue that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S) | h^n] = \mathbb{E}[\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(S) | h^n] = \mu^n(S), \quad (15)$$

provided that $0 \leq \gamma < 2$.

We first argue this in the case $n = 0$. Since Proposition 1.1 implies the existence of the limit, it is enough to show that the random variables $M_\varepsilon = \mu_\varepsilon(S)$ are uniformly integrable as $\varepsilon \rightarrow 0$. Let $M = \mathbb{E}M_\varepsilon$ for ε small enough so that $B_\varepsilon(S) \subset D$. (By (9) this expectation is the same for all sufficiently small ε .) The uniform integrability is equivalent to the statement that the probability measures $\eta_\varepsilon := M^{-1}M_\varepsilon dM_\varepsilon$ are tight, i.e., for all δ there exists a constant $C > 0$ such that $\eta_\varepsilon([C, \infty)) < \delta$ for all ε . (Here $M^{-1}M_\varepsilon dM_\varepsilon$ denotes the probability measure on \mathbb{R} whose Radon-Nikodym derivative with respect to the law of M_ε is given by $M^{-1}M_\varepsilon$.) Since M_ε is a function of h , this is equivalent to the statement that with respect to the measure $M^{-1}M_\varepsilon(h)dh$ the random variables $M_\varepsilon(h)$ are tight. This in turn can be rewritten as the statement that for each δ we can find a C such that

$$\Theta_\varepsilon^S\{M_\varepsilon(h) > C\} < \delta$$

for all ε .

Let $\varepsilon_0 = \sup\{\varepsilon' : B_{\varepsilon'}(S) \subset D\}$. We sample the pair (z, h) in the following steps. In the first step, we sample z from its marginal law (which is independent of ε for ε sufficiently small). Write $\tilde{h} = h - \gamma \xi_\varepsilon^z$. Given z , the law of \tilde{h} is that of a GFF on D . In the second step, we sample $\mathcal{B}_t = \tilde{h}_{e^{-t}\varepsilon_0}(z) - \tilde{h}_{\varepsilon_0}(z)$ for all $t \in [0, -\log(\varepsilon/\varepsilon_0)]$. By Proposition 3.3, \mathcal{B}_t is a Brownian motion on this interval independent of z . The conditional expectation of \tilde{h} given

the whole process \mathcal{B}_t (which we have defined only for $t \in [0, -\log(\varepsilon/\varepsilon_0)]$) and z is given by the function (viewed as a distribution)

$$\tilde{h}'(w) := \mathbb{E}[\tilde{h}(w)|z, \mathcal{B}_t] = \begin{cases} \mathcal{B}_{-\log \frac{\max\{|z-w|, \varepsilon\}}{\varepsilon_0}} & |z-w| < \varepsilon_0 \\ 0 & |z-w| \geq \varepsilon_0 \end{cases}.$$

Note that once z is fixed, for each w the mean value of $\tilde{h}'(\cdot)$ on $\partial B_\varepsilon(w)$ (which we denote by $\tilde{h}'_\varepsilon(w)$) is a weighted average of \mathcal{B}_t over values of t between $0 \vee -\log \frac{|w-z|+\varepsilon}{\varepsilon_0}$ and $-\log \frac{|w-z|-\varepsilon}{\varepsilon_0} \wedge -\log \frac{\varepsilon}{\varepsilon_0}$. From this it is not hard to see that given z the variance of $\tilde{h}'_\varepsilon(w)$ is between these two values of t . We claim that these bounds differ from each other by at most the additive constant $\log 3$. Exponentiating the bounds and multiplying by ε_0 , this is equivalent to the statement that $\varepsilon_0 \wedge (|w-z| + \varepsilon)$ differs from $(|w-z| - \varepsilon) \vee \varepsilon$ by a multiplicative factor of at most three (recall that $\varepsilon < \varepsilon_0$), which is easily checked. Thus the variance of $\tilde{h}'_\varepsilon(w)$ is within $\log 3$ of the former bound

$$u = u(w) = 0 \vee -\log \frac{|w-z| + \varepsilon}{\varepsilon_0}.$$

The conditional variance of $\tilde{h}_\varepsilon(w)$, given $\tilde{h}'_\varepsilon(w)$, is within $\log 3$ of its initial variance minus this value. Thus, with respect to Θ_ε^S , we have

$$\mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(w)} | z, \mathcal{B}_t] \asymp \exp \left(\gamma \tilde{h}'_\varepsilon(w) + \gamma^2 \xi_\varepsilon^z(w) - \gamma^2 u(w)/2 \right) \asymp \exp \left(\gamma \tilde{h}'_\varepsilon(w) + \gamma^2 u(w)/2 \right),$$

where \asymp indicates equality up to a multiplicative factor bounded between positive constants uniformly in ε and z .

Now, given any positive constants a and b , there is a positive probability that a Brownian motion \mathcal{B}_t run for an infinite amount of time will satisfy $\gamma \mathcal{B}_t < a + bt$ for all $t \geq 0$. In fact, for each fixed b , this probability can be made as close to one as possible by taking a sufficiently large. Since $0 \leq \gamma < 2$ we can choose a value of b with $0 < b < 2 - \gamma^2/2$. Then note that when $\gamma \mathcal{B}_t < a + bt$ for all t we have, for some constant C_0

$$\mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(w)} | z, \mathcal{B}_t] \leq C_0 e^a |z-w|^{-b-\gamma^2/2},$$

for $|z-w| < \varepsilon_0$, which in turn implies that

$$\begin{aligned} \mathbb{E}[\mu_\varepsilon(S) | z, \mathcal{B}_t] &\leq \int_{B_{\varepsilon_0}(z)} C_0 e^a |z-w|^{-b-\gamma^2/2} dw + \mathbb{E}[\mu_\varepsilon(S \setminus B_{\varepsilon_0}(z))] \\ &\leq \int_{B_{\varepsilon_0}(z)} C_0 e^a |z-w|^{-b-\gamma^2/2} dw + \mathbb{E}[\mu_\varepsilon(S)], \end{aligned}$$

and since $b + \gamma^2/2 < 2$ (and $\mathbb{E}[\mu_\varepsilon(S)]$ is constant for sufficiently small ε), the right hand side is at most a finite constant $C_1 = C_1(a)$ that is independent of ε . Now, given b and a constant $\delta > 0$ we can choose a large enough so that the probability that $\gamma \mathcal{B}_t < a + bt$ for all t is at least $1 - \delta/2$. Then we take $C = \frac{C_1(a)}{\delta/2}$. If there were probability at least δ that $\mu_\varepsilon(S) > C$ then there would have to be probability at least $\delta/2$ that $\gamma \mathcal{B}_t < a + bt$ for all t

and $\mu_\varepsilon(S) > C$, which could contradict our bound on the conditional expectation of $\mu_\varepsilon(S)$ given that $\gamma\mathcal{B}_t < a + bt$ for all t . This implies that the probability measures η_ε are tight, which in turn completes the proof of (15) in the case $n = 0$ and $h^0 = 0$.

To extend to the case $n \neq 0$, note that since the random variables $\mu_\varepsilon(S)$ almost surely converge to a limit (with expectation $\lim_{\varepsilon \rightarrow 0} \mathbb{E}\mu_\varepsilon(S)$), it must be the case that conditioned on h^n (for almost all h^n), we still have that $\mu_\varepsilon(S)$ almost surely converges to a limit; the fact that

$$\mathbb{E}[\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(S) | h^n] \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S) | h^n]$$

for almost all h^n is immediate from Fatou's lemma, and the fact that equality holds almost surely follows from the unconditional result. The extension of (15) to non-zero h^0 is trivial for functions that are piecewise constant on diadic squares, and the more general case follows easily by approximation by piecewise constant functions.

Proposition 1.2 is an immediate consequence of (14) and (15). \square

4 KPZ proofs

4.1 Circle average KPZ

Fix $z \in D$ and some ε_0 such that $B_{\varepsilon_0}(z) \subset D$. For any $\varepsilon \leq \varepsilon_0$ write $t = -\log(\varepsilon/\varepsilon_0)$ and $V_t = h_\varepsilon(z) - h_{\varepsilon_0}(z)$. The law of V_t is that of a Brownian motion with $V_0 = 0$ (by Proposition 3.3). It follows from Proposition 1.2, and recalling the notation of Proposition 1.1, that the expectation

$$\mathbb{E}_h [\mu_h(B_\varepsilon(z)) | h_\varepsilon(z)] = \mathbb{E}_h \left[\int_{B_\varepsilon(z)} e^{\gamma h} dz | V_t \right]$$

has approximately the form

$$e^{\gamma V_t - \gamma Q t}, \tag{16}$$

in the sense that the ratio of the logarithms of the two quantities tends to 1 as $\varepsilon \rightarrow 0$.

Definition 4.1. Let $\tilde{B}^\delta(z)$ be the largest Euclidean ball in D centered at z for which (16) is equal to δ . The radius of this ball is e^{-T_A} where

$$T_A := \inf\{t : -V_t + Qt = A\},$$

and $A := -(\log \delta)/\gamma$.

As a step towards Theorem 1.5 we prove the following in this section, which is perhaps the most straightforward form of KPZ to prove:

Theorem 4.2. Theorem 1.5 holds with $B^\delta(z)$ replaced with $\tilde{B}^\delta(z)$. That is, in the setting of Theorem 1.5, if

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}\mu_0\{z : B_\varepsilon(z) \in \mathcal{X}\}}{\log \varepsilon^2} = x,$$

then it follows that, when \mathcal{X} and μ are chosen independently, we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu \{z : \tilde{B}^\delta(z) \in \mathcal{X}\}}{\log \delta} = \Delta,$$

where Δ is the non-negative solution to

$$x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta.$$

We present two proofs: the first based on exponential martingales, the second based on large deviations theory and Schilder's theorem. (The first proof is shorter, but readers familiar with large deviations of Brownian motion will recognize that it is essentially the second proof in the disguise.)

Both proofs use the fact that

$$\mathbb{E}_h \mu \{z : \tilde{B}^\delta(z) \in \mathcal{X}\}$$

is proportional to

$$\Theta \{(z, h) : \tilde{B}^\delta(z) \in \mathcal{X}\},$$

to replace an expectation computation with a probability computation. (Recall the definition of Θ from Section 3.3.) While this rephrasing is not strictly necessary for the expectation computation below, it is conceptually quite natural.

We use the definition of V_t given above, and assume that the fixed ε_0 is smaller than the distance from \tilde{D} (recall that this was the compact subset of D in Theorem 1.5) to ∂D .

As mentioned in Section 3.3, the Θ conditional law of h given $z \in D$ is that of the original GFF plus the deterministic function $-\gamma \log |z - y|$. Thus (for z restricted to points of distance at least ε_0 from ∂D) the Θ conditional law of V_t given z is that of $\mathcal{B}_t + \gamma t$, where \mathcal{B}_t evolves as a standard Brownian motion—in particular, z is independent of the process V_t .

Proof. The Θ law of T_A is that of

$$\inf \{t : \mathcal{B}_t + at = A = -(\log \delta)/\gamma\}, \quad a := Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2} > 0, \quad (17)$$

where $(\pm)\mathcal{B}_t$ is standard Brownian motion with $\mathcal{B}_0 = 0$. Since z is independent of T_A , the theorem hypothesis implies that conditioned on T_A , the probability q_A that the ball of radius e^{-T_A} centered at z is in \mathcal{X} is approximately $\exp(-2xT_A)$, in the sense that the ratio of the logs of these two quantities tends to 1 as $T_A \rightarrow \infty$. Computing the expectation

$$\mathbb{E} [\exp(-2xT_A)], \quad (18)$$

with respect to a random T_A will give us upper and lower bounds on q_A since it easily follows that

$$\mathbb{E} [\exp(-2x_1T_A)] \leq q_A \leq \mathbb{E} [\exp(-2x_2T_A)], \quad (19)$$

for any fixed $0 < x_2 < x < x_1$ and sufficiently large A .

To compute (18), consider for any β the exponential martingale $\exp(\beta\mathcal{B}_t - \beta^2t/2)$. At the stopping time T_A

$$\mathbb{E} [\exp(\beta\mathcal{B}_{T_A} - \beta^2T_A/2)] = 1.$$

By definition $\mathcal{B}_{T_A} = A - aT_A$. Thus,

$$\mathbb{E} \exp[-(\beta a + \beta^2/2)T_A] = \exp(-\beta A).$$

Setting $2x := \beta a + \beta^2/2$, we obtain

$$\mathbb{E} \exp(-2xT_A) = \exp(-\beta A). \quad (20)$$

Now if we set $\Delta = \beta/\gamma$, and $a = Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2}$, we find that the equation $2x := \beta a + \beta^2/2$ is equivalent to the KPZ formula. The continuity of this expression and (19) together yield the theorem. \square

We remark that the above yields the explicit probability distribution $P_A(t)$. The inverse Laplace transform $P_A(t)$ of $f_A(x) := \mathbb{E} \exp(-2xT_A)$, with respect to $2x$, is the probability density such that $P_A(t)dt := \text{Prob}(T_A \in [t, t+dt])$. Its explicit expression is

$$P_A(t) = (2\pi)^{-1/2} A t^{-3/2} \exp \left[-(1/2) (A t^{-1/2} - a t^{1/2})^2 \right], \quad (21)$$

where as above we have $A = -(\log \delta)/\gamma$, $t = -\log \varepsilon$ and $a = Q - \gamma$.

4.2 Large deviations proof of circle average KPZ

In this section, we present an alternative proof of Theorem 4.2, using Schilder's theorem.

Lemma 4.3. *Fix a constant $a > 0$. Let \mathcal{B}_t be a standard Brownian motion. For each $A > 0$, write*

$$T_A = \inf\{t : \mathcal{B}_t + at = A\}. \quad (22)$$

Then the family of random variables $A^{-1}T_A$ satisfies a large deviations principle with speed A and rate function

$$I(\eta) = \eta \left(\frac{1}{\eta} - a \right)^2 / 2 = \eta^{-1}/2 - a + a^2\eta/2.$$

Proof. Schilder's Theorem (see Theorem 5.3.2 of [DZ]) gives an LDP for the sample path of $\alpha^{-1}\mathcal{B}_t$ (where \mathcal{B}_t is standard Brownian motion) with speed α^2 and rate function given by the Dirichlet energy. The variable $A^{-1}T_A$ can be written as $\inf\{t : W_t + at = 1\}$ where $W_t = \mathcal{B}_{At}/A$, which has the same law as $\sqrt{A^{-1}}\mathcal{B}_t$. Clearly, among all functions $\phi \in H_1([0, \infty))$ satisfying $\phi(0) = 0$ and $\inf\{t : \phi(t) + at = 1\} \leq \eta$, the one with minimal Dirichlet energy is given by

$$\phi(t) = \begin{cases} (\frac{1}{\eta} - a)t & t < \eta \\ (\frac{1}{\eta} - a)\eta & t \geq \eta. \end{cases}$$

By the contraction principle (Theorem 4.2.1 of [DZ]), the rate function desired in Lemma 4.3 is given by this minimal Dirichlet energy, i.e., $I(\eta) = \eta(\frac{1}{\eta} - a)^2/2$. \square

Lemma 4.4. *Consider the following two part experiment. First choose T_A as above. Then toss a coin that comes up heads with probability*

$$e^{-2xT_A}.$$

Then the probability that the coin comes up heads decays exponentially in A at rate β where β and x are related by

$$\beta = \inf_{\eta} \{I(\eta) + 2x\eta\}, \quad (23)$$

or equivalently by

$$4x = \beta^2 + 2a\beta. \quad (24)$$

Proof. The exponential decay with the exponent given in (23) is an immediate consequence of Varadhan's integral lemma (Theorem 4.3.1 of [DZ]). To derive (24) from (23), we set the derivative of $I(\eta) + 2x\eta$ to zero and find $-\eta^{-2}/2 + a^2/2 + 2x = 0$. Hence the minimum is achieved at

$$\eta_0 = (a^2 + 4x)^{-1/2}. \quad (25)$$

We then compute $\beta = I(\eta_0) + 2x\eta_0$ to be

$$(a^2 + 4x)^{1/2}/2 - a + a^2(a^2 + 4x)^{-1/2}/2 + 2x(a^2 + 4x)^{-1/2}.$$

Simplifying, we have $\beta = (a^2 + 4x)^{1/2} - a$, which is equivalent to (24). \square

Proof of Theorem 4.2. As above, we aim to show that $P\{\tilde{B}^\delta(z) \in \mathcal{X}\}$ scales as $e^{-\beta A} = \delta^{\beta/\gamma} = \delta^\Delta$ where $\Delta = \beta/\gamma$, where δ and ε are related via the stopping time T_A (17). Rescaling T_A by A^{-1} as in (22) puts us in the framework of large deviations Lemma 4.3. As above, to describe the probability $P\{\tilde{B}^\delta(z) \in \mathcal{X}\}$ we can imagine that we first choose the radius ε of $\tilde{B}^\delta(z)$ and then toss a coin that comes up heads with probability ε^{2x} to decide whether the ball is in \mathcal{X} . This puts us in the framework of the second large deviations Lemma 4.4. Using (24), we have

$$4x = \beta^2 + 2a\beta = (\gamma\Delta)^2 + 2a\gamma\Delta,$$

where $a = Q - \gamma$. Plugging in this value of a and simplifying, we obtain the KPZ relation

$$x = \frac{1}{4} (\gamma^2 \Delta^2 + 2\gamma(Q - \gamma)\Delta) = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta.$$

As in the previous proof, if the probability given ε is not exactly ε^{2x} , but the ratio of the log of this probability to the log of ε^{2x} tends to 1 as $\varepsilon \rightarrow 0$, we obtain the same theorem by using alternate values of x to give upper and lower bounds. \square

The optimum $\eta_0 = (a^2 + 4x)^{-1/2}$ obtained in (25) has a natural interpretation — it suggests that (in the large deviations sense described above) T_A/A is concentrated near η_0 .

Equivalently, since $\Delta = \frac{\beta}{\gamma} = \frac{(a^2 + 4x)^{1/2} - a}{\gamma}$, we can say that A/T_A is concentrated near $\gamma\Delta + a = \gamma\Delta + Q - \gamma$, which implies that $\frac{\log \delta}{\log \varepsilon}$ is concentrated near $\gamma(\gamma\Delta + Q - \gamma)$. Note that the same result can also be obtained directly from the explicit probability density (21). This is the concentration one obtains at an α -thick point of the GFF h , where

$$\alpha = \gamma - \gamma\Delta. \quad (26)$$

Very informally, this suggests the quantum support of a quantum fractal of dimension Δ is made up of α -thick points of h . This generalizes the idea of Proposition 3.4, which concerns the case $\Delta = 0$.

4.3 Probability total mass is very small

Lemma 4.5. *Let $\mathbb{D} = B_1(0)$ be the unit disc and fix $\gamma \in [0, 2)$ and take $\mu = e^{\gamma h(z)} dz$ as defined previously. Then the random variable $A = \log \mu(B_{1/2}(0))$ satisfies $p_A(\eta) := \mathbb{P}[A < \eta] < e^{-C\eta^2}$ for some fixed constant $C > 0$ and all sufficiently negative values of η .*

Proof. Let h' be the projection of h onto the space of functions in $H(\mathbb{D})$ that are harmonic inside the two discs $B_{1/4}(1/4)$ and $B_{1/4}(-1/4)$. Recall that the orthogonal complement of this space is the space of functions supported on these discs, or more precisely, the space $H[B_{1/4}(1/4) \cup B_{1/4}(-1/4)]$. Hence, the law of $h - h'$ is that of a sum of independent Gaussian free fields on $B_{1/4}(1/4)$ and $B_{1/4}(-1/4)$ with zero boundary conditions.

Let \underline{h} be the infimum of h' over the union of the two smaller discs $B_- = B_{1/8}(-1/4)$ and $B_+ = B_{1/8}(1/4)$. Write $A_- = \log \mu_{h-h'}(B_-)$ and $A_+ = \log \mu_{h-h'}(B_+)$. By Proposition 2.1 the law of each of A_+ and A_- is the same as the law of $A + \gamma Q \log(1/4) = A - \gamma Q \log 4$; clearly A_+ and A_- are independent of one another. Also, $\mu(B_+) \geq e^{\gamma \underline{h}} \mu_{h-h'}(B_+)$ (and similarly for B_-), which implies

$$A \geq \max\{A_-, A_+\} + \gamma \underline{h}. \quad (27)$$

First we will show that the probability distribution of \underline{h} has superexponential decay. Since h' is harmonic on B_+ (with $h'(1/4) = h'_{1/8}(1/4)$) this h' is the real part of an analytic function on B_+ . In particular, h' restricted to B_+ can be expanded as $h'(1/4) + \sum_{n=1}^{\infty} \operatorname{Re}[a_n 4^n (z - 1/4)^n]$ for some complex a_n . Since each of the random variables $\operatorname{Re} a_n$ and $\operatorname{Im} a_n$ is a real-valued linear functional of h' , it is a Gaussian random variable.

Let F be any linear functional on subspace $H'(D) \subset H(D)$ of functions that are harmonic on B_+ ; then $F(h)$ can be written as $(h, f)_{\nabla}$ for some $f \in H'(D)$ and the variance of $F(h)$ is $(f, f)_{\nabla}$. Note that $(f, f)_{\nabla}^{-1}$ is the smallest Dirichlet energy obtained by F on the set of functions $\{g \in H'(D) : F(g) = 1\}$ (since f has minimal energy on the set $\{g \in H'(D) : F(g) = (f, f)_{\nabla}\}$).

In the case of the linear functional $\operatorname{Re} a_n$ (the case $\operatorname{Im} a_n$ is similar), this minimal Dirichlet energy is at least the Dirichlet energy of $\operatorname{Re} 4^n (z - 1/4)^n$ restricted to $B_{1/4}(1/4)$. (If for any $g \in H(D)$, the linear functionals $\operatorname{Im} a_n$ or $\operatorname{Re} a_m$ for $m \neq n$ applied to g are non-zero, then the Dirichlet energy of g restricted to $B_{1/4}(1/4)$ will be greater than if they were zero, by orthogonality of $\operatorname{Re} z^n$ and $\operatorname{Im} z^m$ on \mathbb{D} ; and the total Dirichlet energy of g on \mathbb{D} is at least the Dirichlet energy of the restriction to $B_{1/4}(1/4)$.) By conformal invariance of the Dirichlet inner product, this energy is given by

$$\int_{\mathbb{D}} n^2 |z^{n-1}|^2 dz = n^2 \int_0^1 r^{2n-2} 2\pi r dr = \pi n.$$

This implies that the variances of $\operatorname{Re} a_n$ and $\operatorname{Im} a_n$ are at most $1/(\pi n)$.

In particular, the variance of $|a_n| r^n$, for any fixed $r < 1$, will decay exponentially in n . Thus, the probability that even one of the a_n satisfies $|a_n| r^n > c$, where c is a fixed constant, decays quadratic-exponentially in c . It follows that the probability density function \underline{p} of \underline{h} satisfies $\underline{p}(\eta) < e^{-\underline{C}\eta^2}$ for some $\underline{C} > 0$ and all sufficiently negative η .

Now, let $P_1(\eta)$ be the probability that $\underline{h} < .1\eta/\gamma$ and $A < \eta$. Let $P_2(\eta)$ be the probability that $A < \eta$ and $\underline{h} \geq .1\eta/\gamma$. Then $p_A(\eta) = \mathbb{P}[A < \eta] = P_1 + P_2$. From the above discussion, we have $P_1(\eta) \leq e^{-\underline{C}\eta^2}$ for all sufficiently negative values of η . Note from (27) that

$$P_2(\eta) \leq [p_A(.9\eta + \gamma Q \log 4)]^2$$

and

$$P_2(\eta) \leq [P_1(.9\eta + \gamma Q \log 4) + P_2(.9\eta + \gamma Q \log 4)]^2 \leq [e^{-C'\eta^2} + P_2(.9\eta + \gamma Q \log 4)]^2,$$

for some C' . Fix a sufficiently negative η_0 and inductively determine η_k via $\eta_{k-1} = .9\eta_k + \gamma Q \log 4$. The above can be stated (with a modified C') as

$$P_2(\eta_k) \leq \left(e^{-C'\eta_{k-1}^2} + P_2(\eta_{k-1}) \right)^2.$$

If we write $p_k = \frac{P_2(\eta_k)}{e^{-C'\eta_k^2}}$, then this can be restated as $p_k \leq (1 + p_{k-1})^2 e^{-2C'(\eta_{k-1}^2 - \eta_k^2/2)}$. It is easy to see that we can have $p_k > 1$ for only finitely many k , which implies that the lemma holds when restricted to the sequence η_k . Because of the monotonicity of $p_A(\eta)$, this implies the lemma for all η . \square

From the lemma above, it is easy to derive the following, which includes a restatement of (16), together with a strong upper bound on the probability that $\mu(B_\varepsilon(z))$ is much lower than this expectation.

Lemma 4.6. *Fix z and ε so that $B_\varepsilon(z) \subset D$. Then*

$$\mathbb{E}[\mu(B_\varepsilon(z)) | h_\varepsilon(z)] \asymp \varepsilon^{\gamma Q} e^{\gamma h_\varepsilon(z)},$$

where

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2},$$

as in Proposition 2.1. Moreover, conditioned on $h_{\varepsilon'}(z)$, for all $\varepsilon' \geq \varepsilon$, we have that

$$\mathbb{P} \left[\frac{\mu(B_\varepsilon(z))}{\varepsilon^{\gamma Q} e^{\gamma h_\varepsilon(z)}} < e^{-t} \right] \leq C_1 e^{-C_2 t^2},$$

for some positive constants C_1 and C_2 independent of $t \geq 0$, z , D , and the values $h_{\varepsilon'}(z)$ for $\varepsilon' \geq \varepsilon$.

Roughly speaking, the above lemma says that the total quantum mass in a ball is unlikely to be a lot smaller than the mass we would predict given the average value of h on the boundary of that ball; the following says that (even when we use the Θ measure), the total quantum mass has some constant probability to be (at least a little bit) smaller than this prediction.

Lemma 4.7. *Let z and h be chosen from Θ^S for a fixed compact subset S of D , and fix a $\delta > 0$, with quantum balls $B^\delta(z)$ and $\tilde{B}^\delta(z)$ defined as in Definition 1.3 and Definition 4.1. Conditioned on the event $\tilde{B}^\delta(z) \subset \tilde{D}$ and on the radius of $\tilde{B}^\delta(z)$, the conditional probability that $\tilde{B}^\delta(z) \subset B^\delta(z)$ is bounded below by a positive constant independent of D , \tilde{D} , and δ .*

4.4 Proof of interior KPZ

In this section we derive Theorem 1.5 as a consequence of Lemma 4.6, Lemma 4.7, and the arguments in Theorem 4.2.

Proof of Theorem 1.5. We use the same notation as in Theorem 4.2, but we write $\bar{T}_A = -\log \bar{\varepsilon}$ where $\bar{\varepsilon}$ is the radius of $B^\delta(z)$. The proof of the Theorem 4.2 carries through exactly once we show that

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E} [\exp (-2x \bar{T}_A)]}{\log \mathbb{E} [\exp (-2x T_A)]} = 1. \quad (28)$$

Lemma 4.6, applied for $t/\gamma = A(1-a)$ and for a fixed $a < 1$, implies that given T_A , the probability that $\bar{T}_A < T_{aA}$ decays superexponentially in A for any $a < 1$. This implies that

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E} [\exp (-2x \bar{T}_A)]}{\log \mathbb{E} [\exp (-2x T_{aA})]} \leq 1,$$

and since this holds for all $a < 1$, the result follows immediately from Lemma 4.7 and the continuity of the coefficient of A in the exponent in (20). \square

5 Box formulation of KPZ

In this section we prove Proposition 1.6.

Proof of Proposition 1.6. The first fact is standard; observe that if ε is a power of two then $S_\varepsilon(X) \subset B_{2\varepsilon}(X)$, since the ball of radius 2ε about a point contains any diadic box of width ε that contains the same point. Similarly, $B_{2\varepsilon}(z)$ is contained in the union of a diadic box — of width 2ε , containing z — with the eight diadic boxes of the same size whose boundaries touch its boundary. This implies that $B_{2\varepsilon}(X)$ is contained in the union of $S_{2\varepsilon}(X)$ and corresponding 8 translations of $S_{2\varepsilon}(X)$, so $\mu_0(B_{2\varepsilon}(X)) \leq 9\mu_0(S_{2\varepsilon}(X))$.

For the second part, we first prove the statement where we replace $\delta N(\mu, \delta, X)$ with the total mass of the set $S^\delta(X)$ of points in the diadic boxes of quantum area at most δ that intersect X . In this case, the proof is similar to that of Theorem 4.2. We use the notation of Theorem 4.2 but set \tilde{T}_A to be $-\log \varepsilon$ where ε is the largest value for which the diadic box with edge length ε has μ area at most δ . The remainder of the argument is essentially the same as the proof of Theorem 1.5. We need only replace \bar{T}_A with \tilde{T}_A and note that

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E} [\exp (-2x \tilde{T}_A)]}{\log \mathbb{E} [\exp (-2x T_A)]} = 1,$$

by the same argument used to compare T_A and \bar{T}_A .

Having obtained this result, it remains only to show that

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \delta N(\mu, \delta, X)}{\log \mathbb{E} \mu(S^\delta(X))} = 1.$$

The lim inf is clearly at most 1, since by definition, the denominator is greater or equal to the numerator. The other direction uses the same argument at the end of the proof of Theorem 1.5. It suffices to note that if we fix a constant $a < 1$, Lemma 4.6 implies that given T_A , the probability that $\tilde{T}_A < T_{aA}$ decays superexponentially in A for any fixed $a < 1$. \square

6 Boundary KPZ

Most of the results in this paper about random measures on D have straightforward analogs about random measures on ∂D . The proofs are essentially identical, but we will sketch the differences in the arguments here.

Suppose that D is a domain with piecewise linear boundary and that h is an instance of the GFF on D with free boundary conditions, normalized to have mean zero.

This means that $h = \sum_n \alpha_n f_n$ where the α_n are i.i.d. zero mean unit variance normal random variables and the f_n are an orthonormal basis, with respect to the inner product

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

of the Hilbert space closure $H(D)$ of the space of C^∞ bounded real-valued (but *not* necessarily compactly supported) functions on D with mean zero.

Note that if f is a compactly supported smooth function on D for which $-\Delta f = \rho$, then integration by parts implies that the variance of (h, ρ) is the Dirichlet energy of f —same as in the zero boundary case. Similarly, suppose that f is a smooth function that is *not* compactly supported but has a gradient that vanishes in the normal direction to ∂D , and we write $\rho = -\Delta f$. Then integration by parts implies that the variance of (h, ρ) is $(f, f)_\nabla$.

We can also make sense of $h_\varepsilon(z)$, for a point z on the boundary of D , to be the mean value of z on the semicircle of radius ε centered at z and contained in the domain D . In this case, if ε_0 is small enough so that B_{ε_0} contains no corners of D and exactly one semi-disc of B_{ε_0} lies in D , we have that $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is a standard Brownian motion in time $2t = -2 \log(\varepsilon/\varepsilon_0)$, as in Proposition 3.3. The $2t$ in place of t comes from two factors: first, by integration by parts, $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is equal to $(h, \xi)_\nabla$, where $\xi(\cdot)$ is the continuous function which is equal to $2 \log |z - \cdot|$ on the half-annulus $\mathbb{H} \cap \{y : \varepsilon < |y - z| < \varepsilon_0\}$ and is constant outside of the half-annulus. (The $2 \log |z - \cdot|$ in place of $\log |z - \cdot|$ comes from the fact that we are taking an average over half a circle.) The variance of $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is given by the Dirichlet energy $(\xi, \xi)_\nabla$, which is twice as big as before (given the factor of 2 in the definition of ξ and the fact that the integral is only over half as much area).

Thus at a point on the interior of one of the boundary lines of D , the variance of $h_\varepsilon(z)$ scales like $-2 \log \varepsilon$ instead of $-\log \varepsilon$. We define the boundary measure $\mu_\varepsilon^B := \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(z)/2} dz$, where in this case dz is Lebesgue measure on the boundary of D . Here we use $e^{\gamma h_\varepsilon(z)/2}$ instead of $e^{\gamma h_\varepsilon(z)}$ because we are integrating a length instead of an area; as before, the power of ε chosen makes the factor preceding dz an exponential martingale in time $t = -\log \varepsilon$.

We define μ^B to be the weak limit as $\varepsilon \rightarrow 0$ of the measures μ_ε^B (see the theorem below for existence of this limit when $0 \leq \gamma < 2$). For $z \in \partial D$ we write $\hat{B}_\varepsilon(z) := B_\varepsilon(z) \cap \partial D$ and we define $\hat{B}^\delta(z)$ to be the (largest) set $\hat{B}_\varepsilon(z)$ whose μ^B measure is δ .

Likewise define

$$\hat{B}_\varepsilon(X) = \{z \in \partial D : \hat{B}_\varepsilon(z) \cap X \neq \emptyset\}$$

and

$$\hat{B}^\delta(X) = \{z \in \partial D : \hat{B}^\delta(z) \cap X \neq \emptyset\}.$$

We say that a (deterministic or random) fractal subset X of the boundary of D has **Euclidean expectation dimension** $1 - \tilde{x}$ and **Euclidean scaling exponent** \tilde{x} in the bound-

ary sense if the expected measure of $\hat{B}_\varepsilon(X)$ decays like $\varepsilon^{\tilde{x}}$, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E} \mu_0(\hat{B}_\varepsilon(X))}{\log \varepsilon} = \tilde{x}.$$

We say that X has **boundary quantum scaling exponent** $\tilde{\Delta}$ if when X and μ^B (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu^B(\hat{B}^\delta(X))}{\log \delta} = \tilde{\Delta}.$$

Theorem 6.1. *Given the assumptions above, Proposition 1.1 and Theorems 1.5 and 4.2 hold, precisely as stated, when μ_ε is replaced by μ_ε^B , μ is replaced by μ^B ; μ_0 (Lebesgue measure on D) is replaced by Lebesgue measure on ∂D ; B_ε and B^δ are replaced with \hat{B}_ε and \hat{B}^δ , respectively; and the compact subset of D is replaced with a closed subinterval of one of the boundary line segments of D .*

Proof. The proofs in the boundary case proceed exactly the same as in the interior point case, up to factors of 2 in various places. We sketch the proof of an analog of Theorem 4.2 in order to indicate where those factors of 2 appear.

Write $t = -\log(\varepsilon/\varepsilon_0)$, and let $V_t = h_\varepsilon(z) - h_{\varepsilon_0}(z)$. It is not hard to see that the expectation of the boundary line integral

$$\mathbb{E}_h \left[\mu_h^B(\hat{B}_\varepsilon(z)) | h_\varepsilon(z) \right] = \mathbb{E}_h \left[\int_{\hat{B}_\varepsilon(z)} e^{\gamma h/2} dz | V_t \right]$$

has approximately the form (which replaces (16))

$$\exp \left[\frac{\gamma}{2} V_t - \frac{\gamma}{2} Q t \right], \quad (29)$$

in the sense that the ratio of the logs of the two quantities tends to 1. Let $\tilde{B}^\delta(z)$ now be the largest Euclidean ball $B_\varepsilon(z)$ in D centered at $z \in \partial D$ for which (29) is equal to the quantum length δ .

As before, the Θ conditional law of h given z is that of the original GFF plus the deterministic function $-\gamma \log |z - y|$ (minus a bounded function of y).

Then given $z \in \partial D$, the Θ conditional law of V_t is that of $\mathcal{B}_{2t} + \gamma t$, where \mathcal{B}_{2t} evolves as a Brownian motion with twice the variance of standard Brownian motion, because of the free boundary conditions on ∂D . Using (29), we have

$$\mathbb{E}_h \left[\int_{\hat{B}_\varepsilon(z)} e^{\gamma h/2} dz | V_t \right] \asymp \exp \left[\frac{\gamma}{2} \mathcal{B}_{2t} + \frac{1}{2} \gamma^2 t - \frac{\gamma}{2} Q t \right]. \quad (30)$$

This will be equal to the quantum boundary length δ at the smallest t for which $\gamma^2 t + \gamma \mathcal{B}_{2t} - \gamma Q t = 2 \log \delta$. That is, $-\mathcal{B}_{2t} + (Q - \gamma)t = -2(\log \delta)/\gamma$. If we set $A := -(\log \delta)/\gamma$, this smallest time is a stopping time T_A such that

$$T_A = \inf \{ t : \mathcal{B}_{2t} + at = 2A = -2(\log \delta)/\gamma \}, \quad a = Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2} > 0. \quad (31)$$

As above, we consider the two part experiment in which we first sample T_A and then sample z and check to see whether the ball of radius $\varepsilon = e^{-T_A}$ intersects X on the boundary. Given T_A , the ratio of the logarithms of this probability and

$$\mathbb{E} [\exp (-\tilde{x}T_A)]$$

tends to 1 as $A \rightarrow \infty$. Consider next for any β the exponential martingale $\exp \left(\frac{\beta}{2} \mathcal{B}_{2t} - \frac{\beta^2}{4} t \right)$, such that

$$\mathbb{E} \left[\exp \left(\frac{\beta}{2} \mathcal{B}_{2t} - \frac{\beta^2}{4} t \right) \right] = \mathbb{E} \left[\exp \left(\frac{\beta}{2} \mathcal{B}_0 \right) \right] = 1.$$

At the stopping time T_A in particular:

$$\mathbb{E} \left[\exp \left(\frac{\beta}{2} \mathcal{B}_{2T_A} - \frac{\beta^2}{4} T_A \right) \right] = 1.$$

By definition $\mathcal{B}_{2T_A} = 2A - aT_A$. One thus gets the identity

$$\mathbb{E} \exp [-(\beta a/2 + \beta^2/4)T_A] = \exp(-\beta A),$$

and it now suffices to identify $2\tilde{x} := \beta a + \beta^2/2$ to obtain the boundary KPZ with $\tilde{\Delta} := \beta/\gamma$ and

$$\mathbb{E} \exp(-\tilde{x}T_A) = \delta^{\tilde{\Delta}} = \exp(-\beta A) = \exp \left\{ -A[(a^2 + 4\tilde{x})^{1/2} - a] \right\}.$$

□

The reader may observe that the boundary measures described above are preserved under the transformations described in Proposition 2.1. One can use this to define the boundary measure on more general domains, which may not have piecewise linear boundary conditions.

We also remark that a similar procedure to that above allows us to make sense of measure restricted to lines in the interior of the domain.

7 Discrete random surface dimensions and heuristics

Historically, one of the uses of the KPZ formula has been to make heuristic predictions about the scaling exponents of random fractal subsets of the plane (see, e.g., [Dup99b, DFGG00, Dup04, Dup00, Dup06], and the references surveyed therein for much more detail).

In this subsection, we give a very rough and very brief sketch of what such a heuristic might entail in a simple example. Readers familiar with discrete quantum gravity models (a.k.a. random planar map models, random quadrangulation models, etc.) should note that these models have natural interpretations as continuum random metric spaces as well. For example, a random planar quadrangulation M_n on the sphere — chosen uniformly from the set of all simply connected planar quadrangulations with n quadrilaterals — can be viewed as a manifold by endowing each quadrilateral with the metric of a unit square. (Of course, the resulting manifold will have singularities: negative curvature point masses at vertices where more than four unit squares coincide and positive curvature point masses at vertices where fewer than four unit squares coincide.) We may then choose a uniform square from among this set. Taking an “infinite volume limit” (as $n \rightarrow \infty$) one obtains

an infinite random quadrangulation M_∞ with a distinguished square. (See, e.g., [AS03] for a precise description of this construction for triangulations.) This infinite random metric can be conformally mapped to the plane in such a way that the center of the distinguished square is mapped to the origin and the volume of the image of the distinguished square is a constant δ (with a rotation chosen uniformly at random). The images of the unit squares of M_∞ form a tiling of \mathbb{C} by “conformally distorted” unit squares. Different squares have different sizes with respect to the Euclidean metric on the plane; intuitively, one would expect such a tiling to look something vaguely like the tilings in Figures 1, 2, and 3 except that the “squares” would be randomly oriented and distorted. The pullback of the intrinsic metric of M_∞ to the plane via this map takes the form $e^\lambda(dx^2 + dy^2)$ for some function random λ (which has logarithmic singularities at the images of the vertices of the squares). Although the equivalence of Liouville quantum gravity and discrete quantum gravity is taken as an Ansatz throughout much of the literature, to our knowledge the following is the first precise conjecture for the complete scaling limit of a discrete quantum gravity model:

Conjecture 7.1. *As $\delta \rightarrow 0$, the function λ converges in law (e.g., w.r.t. to the weak topology on the space of distributions on the plane modulo additive constants) to $\gamma(h - \gamma \log |\cdot|)$ where h is an instance of the whole plane Gaussian free field (defined up to additive constants) and $\gamma^2 = \kappa = 8/3$.*

We further conjecture that other values of γ are obtained by choosing a random quadrangulation together with a statistical physical model on the quadrangulation (FK cluster model, percolation, $O(N)$ model, uniform spanning tree); in this case, the probability of a given quadrangulation is proportional to the partition function of the statistical physics model on that quadrangulation. (See the references on random matrix theory and geometrical models cited in the introduction for much more detail; see [Dup06] for a review with additional references.) One can also consider scaling limits on spheres or higher genus surfaces, as well as different kinds of marked points (corresponding to different logarithmic singularities in the scaling limit); however, these are a bit more complicated to describe, so we limit attention to the infinite volume case for now.

By the usual conformal invariance Ansatz, it is natural to expect that if one conditions on the infinite quadrangulation, and then samples the loops or trees in these models (as mapped into the plane), their law (in the scaling limit) will be *independent* of the metric.

Now suppose that for each n we define a random subset X_n of M_n (for example, X_n could be the set of the squares hit by a simple random walk started at the root square and stopped the first time that the walk hits a square on the boundary of the quadrangulation). Then one can define a discrete scaling exponent (analogous to the box counting exponent in (3), with δ replaced by n^{-1}) as follows:

$$\Delta_D = \lim_{n \rightarrow \infty} \frac{\log \mathbb{E}(n^{-1} |X_n|)}{\log n^{-1}}.$$

Identifying X_n with its image in a conformal map to, say, \mathbb{D} , one might guess that the random pair (X_n, λ_n) has a scaling limit (X, λ) , where X is a random subset of \mathbb{D} (in our example, it might be a Brownian motion) and λ is some form of the Gaussian free field.

If this is the case, then on a heuristic level, one would expect that the quantum scaling exponent of X is $\Delta = \Delta_D$, since, in the notation of Corollary 1.7, if we write $\delta = n^{-1}$, we would expect that $\mathbb{E}[\delta N(\mu, \delta, X)]$ scales like $\mathbb{E}(n^{-1} |X_n|)$.

In discrete quantum gravity models, it is often possible to compute Δ_D explicitly (and rigorously) using random matrix techniques or tree bijections; it is also often possible to compute γ directly using discrete quantum gravity machinery and so heuristically obtain its value in the continuum limit.

Assuming values for Δ_D and γ — and assuming $\Delta = \Delta_D$ — the KPZ formula gives the Euclidean scaling dimension of X . In many interesting examples, X is a random fractal (a Schramm-Loewner evolution, for example, or the outer boundary of a planar Brownian motion) whose Euclidean scaling dimension might not be immediately obvious otherwise.

Finally, we mention that, in the standard realm of conformal field theory, there exists a precise relation between the central charge $c \leq 1$ of the statistical model coupled to quantum gravity and the value of Liouville parameter, $\gamma = (\sqrt{25 - c} - \sqrt{1 - c}) / \sqrt{6}$, [KPZ88, Dav88b, DK89, Sei90, GM93], as well as the corresponding connection between SLE_κ and Liouville quantum gravity models with $\gamma = \sqrt{\min\{\kappa, 16/\kappa\}}$.

Our result extends the validity of the KPZ relation outside that CFT framework to any value of Liouville parameter $\gamma < 2$, with the Ansatz that the fractal set X and the GFF are sampled independently. A possible interpretation of the KPZ relation in that case would be that it describes the quantum geometry of the given fractal in the *quenched* random metric generated by random graphs, equilibrated with a conformally invariant system with a value of c or κ corresponding to the chosen value of γ . For example, one could first choose a random graph weighted by the critical Ising model partition function; and then perform a loop erased random walk on that graph, ignoring Ising clusters. In this case, one would expect the Euclidean dimension of the path to be that of SLE_2 (which corresponds to loop erased random walk), while the value of γ describing the metric would be $\sqrt{3}$ (which corresponds to the critical Ising model), and one could use KPZ to predict the quantum scaling dimension.

Similar ideas appeared in previous numerical work [ABT99, JJ99], but the data so far appear as inconclusive.

8 Future work

The second author currently has two papers in preparation which aim to give additional support to the conjectures in the previous section. This section contains a brief outline of these forthcoming results. A joint paper from a physics perspective is also in preparation about the relation between Liouville quantum gravity and SLE.

8.1 Quantum wedges and conformal welding

In this section, we assume that $0 < \gamma < 2$ has been fixed. Recall from Proposition 2.1 that two pairs (D, h) and (\tilde{D}, \tilde{h}) have the property that the quantum measure on one is the image of the quantum measure on the other under the conformal map $\psi : \tilde{D} \rightarrow D$ if and only if

$$\tilde{h} = h \circ \psi + Q \log |\psi'|. \quad (32)$$

We define a “metric” to be an equivalence class of pairs (D, h) under the following relation: $(D, h) \sim (\tilde{D}, \tilde{h})$ if there exists a conformal map ψ for which (32) holds. This

is a metric in the sense that areas and lengths can be computed via the definitions of this paper. We define a “metric with k marked points” to be an equivalence class of the set of triples $(D, h, (z_1, z_2, \dots, z_k))$ under the following relation: $(D, h, (z_1, z_2, \dots, z_k)) \sim (\tilde{D}, \tilde{h}, (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_k))$ if there exists a conformal map ψ for which (32) holds and $\psi(\tilde{z}_i) = z_i$ for each $1 \leq i \leq k$.

Roughly speaking, a **quantum wedge** is the metric obtained by taking h to be an instance of the free boundary Gaussian free field on an infinite wedge $\{z : 0 < \arg z < \theta\}$ for some θ . If we conformally map the half plane to the wedge via the map $\psi_\theta(\tilde{z}) = \tilde{z}^{\theta/\pi}$, then we see that this is also the metric defined by taking $\tilde{D} = \mathbb{H}$ and letting \tilde{h} be an instance of the free boundary Gaussian free field on \mathbb{H} *plus* the deterministic function $Q \log |\psi'_\theta|$ which up to additive constant is $Q \log |\tilde{z}^{\theta/\pi-1}| = Q(\theta/\pi - 1) \log |\tilde{z}| = -\alpha \log |\tilde{z}|$ for $\alpha := Q(1 - \theta/\pi)$. Our precise definition of quantum wedge (below) will include any $\alpha \in \mathbb{R}$ less than Q (which is the limit as the wedge angle tends to zero; note that our definition will allow for wedge angles greater than 2π).

Now, we have to be a bit careful since h is only defined up to additive constant, and adding a constant to h (thereby “rescaling the metric by a constant”) does not yield an equivalent metric. However, if we define $\psi^b(z) = bz$, for some $b > 0$, then

$$\tilde{h} = h \circ \psi^b + Q \log |(\psi^b)'| = h \circ \psi^b + Q \log b$$

does yield an equivalent metric. Thus, rather than defining h modulo additive constants, we choose h modulo transformations of this form. (To be more precise, note that under the map $\psi^b(z)$ the function $\mathcal{B}_t = h_{e^{-t}}$ transforms as $\mathcal{B}_t \rightarrow \mathcal{B}_{t-\log b} + Q \log b$. In this case, the graph (t, \mathcal{B}_t) in the t and \mathcal{B}_t plane is translated along the span of the vector $(-1, Q)$. Two \mathcal{B}_t functions can arise from the same metric—and may be considered “equivalent”—if one is obtained from the other by a translation of the plane preserving the span of $(-1, Q)$. We can define a canonical representative of the equivalence class by translating along the line so that \mathcal{B}_t first hits this line at the origin. In a quantum wedge, the law of \mathcal{B}_t for positive t is simply the law of Brownian motion with the given drift and \mathcal{B}_t for negative t is the same but conditioned not to hit the line before 0. The law of the difference between h and its expectation given \mathcal{B}_t turns out to be well defined without any undetermined additive constants.)

The result is a random metric with two marked points (0 and ∞) whose law is invariant under constant rescalings (Möbius transformations fixing those two points). Each quantum wedge—when parameterized by \mathbb{H} as discussed above—has an infinite amount of quantum mass, almost surely, but only a finite amount corresponding to any particular bounded subset of \mathbb{H} . (In particular, the law of a quantum wedge is not symmetric under reversing the two marked points, since every neighborhood of its second point has infinite mass, and this is not true of the first point.) The **weight** of the quantum wedge is the number defined from α as follows: $W := \gamma(\gamma + \frac{2}{\gamma} - \alpha)$. Taking α less than Q corresponds to taking $W > \gamma(\gamma + \frac{2}{\gamma} - Q) = \gamma^2/2$. This definition is motivated by the following:

Theorem 8.1. *Choose a quantum wedge \mathcal{W} of positive weight W , represented by some $(D, h, (z_1, z_2))$. Suppose $W = W_1 + W_2$ for some $W_i > \gamma^2/2$ and then independently choose an $\text{SLE}_{\rho_1, \rho_2, \kappa}$, for $\rho_i = W_i - 2$ and $\kappa = \gamma^2$, from z_1 to z_2 . Let Γ denote the set of points on the curve and let D_1 and D_2 denote left and right components of $D \setminus \Gamma$. (The condition*

$W_i > \kappa/2$ corresponds to the condition that the ρ_i are in the range for which the path almost surely does not intersect the boundary [LSW03].)

Then the random metrics $\mathcal{W}_1 = (D_1, h, (z_1, z_2))$ (with h restricted to D_1) and $\mathcal{W}_2 = (D_2, h, (z_1, z_2))$ (with h restricted to D_2) are independent. Each \mathcal{W}_i has the law of a quantum wedge with weight W_i .

In fact, one can use Theorem 8.1 to give a definition of quantum wedges of positive weight less than $\gamma^2/2$; these wedges are not wedges topologically (since their left side hits their right side at a random fractal set of points) but they can be well defined, and the above theorem holds for these wedges as well. It also turns out that \mathcal{W} is uniquely determined by the \mathcal{W}_i and may be obtained by *conformal welding* the right side of \mathcal{W}_1 to the left side of \mathcal{W}_2 , where each is parameterized by quantum length. (This is closely related to a conjecture due to Peter Jones that SLE can be defined via conformal welding and the boundary measure induced by the Gaussian free field.) If we use the boundary analog of the correspondence in (26) to relate Δ and W , we find that this fact is equivalent to the additivity of quantum exponents predicted and advocated by the first author, which is well motivated in discrete quantum gravity models [Dup98, Dup99b, Dup04]. One can also weld the two sides of a wedge of weight W to each other to obtain a **quantum cone** (which is defined identically to the quantum wedge but with the whole plane instead of the half plane) with $\alpha' = \frac{\alpha}{2} + \frac{1}{\gamma}$. We refer to this object as the **quantum cone of weight W** .

8.2 Scaling limits of FK clusters on random graphs

Consider a domain D with marked points z_1 and z_2 , and sample a pair (h, z) from Θ (so that z is an interior marked point) and consider an instance h of the GFF on D with Dirichlet boundary conditions. Then choose a space-filling $\text{SLE}_{\kappa'}$ path Γ from z_1 to z_2 , with $\kappa' = 16/\kappa = 16/\gamma^2$. (When $4 < \kappa' < 8$, the path should be the space-filling analog of $\text{SLE}_{\kappa'}$ constructed via the exploration trees corresponding to the conformal loop ensemble $\text{CLE}_{\kappa'}$, as advocated in [She].)

Next, take an infinite volume limit by zooming in near the point z . The limiting object is a quantum cone with $\alpha = \gamma$, together with a random space filling path that comes from ∞ , hits z , and then fills up the rest of space. We parameterize the path Γ by \mathbb{R} in such a way that $\Gamma(0) = 0$ and $\Gamma([a, b])$ for each $a < b$ in \mathbb{R} , has quantum measure $b - a$.

It turns out that $\Gamma[0, \infty)$ (the “future”) and $\Gamma((-\infty, 0])$ (the “past”) are independent quantum wedges. In fact, since the whole process is stationary with respect to t , this will imply an independence of increments result: namely that $\Gamma([a, b])$ and $\Gamma([c, d])$ are independent as random metrics when the (a, b) and (c, d) do not overlap.

Denote by L_t the quantum length of the left boundary of $\Gamma((-\infty, t])$ minus the length of the left boundary of $\Gamma((-\infty, 0])$ (both boundaries have infinite length; but since the two boundaries agree outside of some finite region, this can be defined as the difference in lengths of the portions that do not agree). Define R_t similarly. It turns out that the processes $L_t + R_t$ and $L_t - R_t$ are independent Brownian motions in the plane, but run at different speeds; for $4 < \kappa'$, we have

$$\frac{\text{Var}[L_t - R_t]}{\text{Var}[L_t + R_t]} = \cot\left(\frac{\kappa}{8}\pi\right).$$

Note that this ratio ranges from ∞ to 0 as $\kappa = 16/\kappa'$ ranges from 0 to 4. It is less than 1 for $\kappa \in (2, 4)$ and equal to 1 for $\kappa = 2$.

We will also define an exploration path corresponding to a critical FK cluster model with parameter q . Given this relation, the analogously defined (L_n, R_n) for these processes converge to the (L_t, R_t) above under the usual Brownian rescalings. The variance ratios agree if we set $q = 2 + 2 \cos \frac{8\pi}{\kappa'} = 2 + 2 \cos \frac{\kappa\pi}{2}$. This relationship between q and κ' is consistent with a prediction made by the first author [Dup03]. In the case of $\kappa = 2$, this convergence is an immediate consequence of a beautiful bijection by Bernardi, which shows that (L_n, R_n) is a simple random walk on \mathbb{Z}^2 [Ber06]. When $q > 4$, the metrics converge to the continuum random tree (a.k.a. branched polymer).

The results above will show (barring technical difficulties), that the discrete quantum gravity loop models converge to a quantum cone endowed with a conformal loop ensemble, independently of the metric — however, this convergence is only in a very special topology, the topology in which metrics are considered close if the corresponding (L, R) processes (which we call **driving functions**) are close. Strengthening the topology to prove a result like Conjecture 7.1 requires in some sense showing that when two processes have close driving functions, there is a high probability that they are close in the sense that, when both metrics are conformally embedded in the plane, the images of the exploration paths are close. This seems to reduce to proving a kind of fancy “random walk in random environment” uniformization result, which, though intuitively plausible, appears extremely difficult to handle. Another way to strengthen the topology would be try to define a distance function on Liouville quantum gravity and prove convergence with respect to a topology on the set of metric spaces. This appears to be even more difficult, although the recent progress on topological scaling limits of discrete planar maps by Le Gall [LG07] and others may prove helpful in the special case $\kappa = 8/3$, as well as the recent study of geodesics in large planar maps and in the Brownian map [LG08] (see also [BG08a, BG08b]).

Acknowledgments: We thank the IAS/Park City Mathematics Institute, where this work was initiated, the School of Mathematics of the Institute for Advanced Study for its gracious hospitality in successive stays during which this work was done, the ICTP in Trieste, the École de physique des Houches and the Centre de recherches mathématiques de l’Université de Montréal where this work could be completed. B.D. wishes especially to thank Michael and Marta Aizenman and Tom Spencer for their generous hospitality in Princeton. It is also a pleasure to thank Jacques Franchi for suggesting the exponential martingale argument used in Section 4.1. We also thank Omer Angel, Peter Jones, Greg Lawler, Andrei Okounkov, and Oded Schramm for stimulating conversations and email correspondence, and Tom Alberts for comments on a prior draft of this paper.

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